A Multiple Secretary Problem with Switch Costs

by

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Abstract

In this thesis, we utilize probabilistic reasoning and simulation methods to determine the optimal selection rule for the secretary problem with switch costs, in which a known number of applicants appear sequentially in a random order, and the objective is to maximize the sum of the qualities of all hired secretaries over all time. It is assumed that the quality of each applicant is uniformly distributed and any hired secretary can be replaced by a better qualified one at a constant switch cost. A dynamic program is formulated and the optimal selection rule for the single secretary case is solved. An approximate solution is given for the multiple secretary case, in which we are allowed to have more than one secretary at a time. An experiment was designed to simulate the interview process, in which respondents were sequentially faced with random numbers that represent the qualities of different applicants. Finally, the experimental results are compared against the optimal selection strategy.

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Contents

Figures

Tables

Chapter 1. Introduction

1.1 Overview of the Secretary Problem

In decision theory, the secretary problem is that of making selection decisions on a known number of items of random quality which are presented sequentially. The aim is to maximize the probability to selecting the best or to maximize the expected payoffs. The decision to reject, once made is irrevocable.

One classic example is the hiring of a secretary. Job applicants are interviewed in succession and the decision to hire is made immediately after the interview before interviewing the remaining applicants. Applicants who are rejected would no longer be available for hire. Other examples include choosing a spouse from a series of suitors, and online auction problems whereby agents arrive and depart over time.

The secretary problem first appeared as a simple, partly recreational problem in the late 1950's and early 1960's, but made its way into the mathematical community. Because the problem is easy to state and has a striking solution, it attracted the attention of many eminent mathematicians and statisticians. Since then, the problem has been extended and generalized to many areas

1.1.1 Classical Secretary Problem

In the classical secretary problem a strategy is sought to maximize the probability of selecting the best hire among n available applicants. What I term as the classical secretary problem is as follows.

1. A known number of applicants with unknown qualities are to be interviewed one by one in random order, all $n!$ possible orders are equally likely.
2. The interviewer at any time, is able to rank the applicants who had been

interviewed in the order of desirability.

3. As each applicant is presented the interviewer must either accept her, in which case the process stops, or reject her, in which case the next candidate in the sequence is interviewed and the interviewer faces the same choice as before.
4. All the applicants once rejected cannot later be recalled later. The interviewer is satisfied with nothing but the best.

Because the interviewer is never able to go back and select a previously interviewed applicant, he clearly has to balance the risk of stopping too soon and accepting an apparently desirable applicant when in fact a better one is still to come, against that of going on for too long and realizing that the best applicant has already been rejected earlier on.

The state of the process at any time may be described by two numbers (i, r), where i is the number of applicants so far interviewed and r is the relative rank of the ith applicant among the first i applicants with rank 1 being the best in the classic secretary problem. An applicant should be accepted only if he is relatively the best among those who have already been observed, and this applicant is called a *candidate*. Thus the ith applicant is a candidate if and only if $r=1$. If we accept a candidate at stage i, the probability we win (i.e. hire the best secretary) is the same as the probability that the best among the first i applicants is also the best among all applicants. This is basically the probability that the best overall candidate overall appears among the first i applicants, namely i/n. Letting $y(i, r)$ denote the probability of selecting an applicant with relative rank r, then if applicant i is a candidate,

$$y(i,1) = \frac{i}{n} \tag{1.1}$$

Let V_i denote the probability of winning using an optimal rule among rules that pass up the first i applicants. Because the best rule among those that pass up the first $i+1$ applicants is available among the rules that pass up only the first i applicants, it is obvious that $V_i > V_{i+1}$. If the probability to select a candidate at stage i is greater than V_i, then it is optimal to stop there. It is also optimal to stop with a candidate after stage i, because $(i+1)/n > j/n > V_i > V_{i+1}$. Therefore, an optimal rule may be found among the rules of the following form: reject the first $k-1$ applicants and then accept the next relatively best applicant, if such an applicant exists.

The probability of 'win' using this form of optimal rules is:

$$
\begin{aligned}
P(k) &= \sum_{i=k}^{n} P(i^{th} \textit{ is best and is selected}) \\
&= \sum_{i=k}^{n} P(i^{th} \textit{ is best}) \sum_{i=k}^{n} P(i^{th} \textit{ is selected} \mid i^{th} \textit{ is best}) \\
&= \sum_{i=k}^{n} \frac{1}{n}\frac{k-1}{i-1} \\
&= \frac{k-1}{n}\sum_{i=k}^{n}\frac{1}{i-1}
\end{aligned} \qquad (1.2)
$$

The optimal value of k is the value which maximizes the probability of 'win'. Because

$$
\begin{aligned}
& P(k) \geq P(k+1) \\
\Rightarrow\ & \frac{k-1}{n}\sum_{i=k}^{n}\frac{1}{i-1} \geq \frac{k}{n}\sum_{i=k+1}^{n}\frac{1}{i-1} \\
\Rightarrow\ & \sum_{i=k+1}^{n}\frac{1}{i-1} \leq 1
\end{aligned} \qquad (1.3)
$$

The optimal rule is to select the first candidate that appears among applicants from stage k onwards, where

$$
k = min\left\{k \geq 1 : \sum_{i=k+1}^{n}\frac{1}{i-1} \leq 1\right\} \approx min\left\{k \geq 1 : log\left(\frac{n}{k}\right) \leq 1\right\} = min\left\{k \geq 1 : k \geq e^{-1}n\right\}. \qquad (1.4)
$$

Hence, for large n it is approximately optimal to pass up a proportion, $e^{-1} = 36.8\%$ of the applicants and then select the next candidate.

1.1.2 Generalized Secretary Problem

The generalized secretary problem can be obtained by replacing the rather restrictive 4th assumption in the classical secretary problem, which is quite restrictive, with the more general objective function: the interviewer is not only interested in the best applicant.

Let a_i and r_i denote the *i*th applicant's absolute rank and its relative rank among the first *i* applicants respectively. A rule vector $s = (s_1, \ldots, s_n)$ dictates that the interviewer stops at on the first applicant for which $r_i \le s_i$. Obviously, the probability that the interviewer stops at the *i*th applicant, conditional on reaching the applicant, is s_i / j. The interviewer's cutoff for selecting an applicant with a relative rank *r*, denoted i_r, is the smallest value *i* for which $r_i \le s_i$. Usually the cutoff representation will be more convenient. The following table shows an example of stopping rule and its cutoff representation. For instance, the value of i_4 is 7 because 7 is the smallest number of *i* for which $s_i \ge 4$.

s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
0	1	2	2	2	3	4	4
	i_1	i_2			i_3	i_4	
	1	3			6	7	

Table 1. Stopping Rule and Its Cutoff Representation

Given that there is a total of *n* applicants, the probability that the *i*th applicant has an absolute rank of a conditional on having *a* relative rank of r_i is given by (Lindely, 1961):

$$P(A=a \mid R=r_i)=\frac{\binom{a-1}{r-1}\binom{n-a}{i-r}}{\binom{n}{i}}. \tag{1.5}$$

Thus the expected payoff for selecting an applicant with relative rank r_i is:

$$E(\pi_i \mid r_i)=\sum_{a=r_i}^{n} P(A=a \mid R=r_i)\pi(a). \tag{1.6}$$

The expected payoff for making a selection at stage *i* for some stage *i* policy s_i is:

$$E(\pi_i \mid s_i)=\frac{\sum_{j=1}^{s_i} E(\pi_i \mid r_i=j)}{s_i}. \tag{1.7}$$

Denoting the expected payoff for starting at stage *i+1* and then following a fixed threshold rule thereafter by v_{i+1}, the value of v_i is:

$$v_i=\frac{s_i}{i}E(\pi_i \mid s_i)+\left(1-\frac{s_i}{i}\right)v_{i+1}. \tag{1.8}$$

Lindley (1961) solved these by combining numerical search methods with dynamic programming. The expected payoff for following a rule *s* is:

$$E(\pi \mid s)=\sum_{i=1}^{n}\left[\prod_{j=0}^{i-1}\left(1-\frac{s^j}{j}\right)\right]\frac{s^i}{i}E(\pi_i \mid s_i)=v_1. \tag{1.9}$$

The optimal rule s^* is the policy *s* that maximizes Eq. 1.9. Denoting the applicant position at which the search is terminated by *m*, the probability that the interviewer stops at the *i*th applicant is:

$$P(m=i)=\left[\prod_{j=0}^{i-1}\left(1-\frac{s^j}{j}\right)\right]\frac{s^i}{i}, \tag{1.10}$$

and the expected stopping position is

$$E(m)=1+\sum_{i=1}^{n-1}\left[\prod_{j=1}^{i}\left(1-\frac{s^j}{j}\right)\right]. \tag{1.11}$$

1.1.3 Multiple-Choice Secretary Problem

The Multiple-Choice secretary problem is a variation of the classical secretary problem in which the algorithm is allowed to choose m secretaries with the goal of maximizing their sum. Kleinberg (2005) presented an algorithm whose competitive ratio, which is defined as the ratio between its online algorithm's performance and the offline algorithm's performance, is $1-O\sqrt{1/m}$.

The basic algorithm is defined recursively as follows. The classical secretary problem is a special case of the multiple-choice secretary problem in which *m=1*. It is approximately optimal to pass up a proportion $e^{-1}=36.8\%$ of the applicants and then select the next candidate. If *m>1* then select a random sample with k elements from the binomial distribution *B(n, 0.5)*. Among these k elements, recursively apply this algorithm until $\lfloor m/2 \rfloor$ elements have been selected.

The remaining $(m-\lfloor m/2 \rfloor)$ elements are selected from the remaining *(n-k)* applicants. Let the k samples be ordered from the largest to the smallest: $y_1 \geq y_2 \geq \ldots \geq y_k$. Among the *(n-k)* applicants, select every element which is greater than $y_{\lfloor m/2 \rfloor}$ until we have either selected m elements or have seen all applicants.

Let υ be the sum of the m largest elements, Kleinberg (2005) suggests that the elements selected from the first k samples have expected modified value of at least $\left(1-5/\sqrt{k/2}\right)\cdot\left(1-1/2\sqrt{k}\right)\cdot(\upsilon/2)$ and those elements selected from the remaining *(n-k)* applicants have expected modified value of at least $\left(1/2-\sqrt{1/k}\right)\upsilon$. Thus the expected modified value of all elements selected by this algorithm is greater than $\left(1-5/\sqrt{k}\right)\upsilon$.

1.1.4 Other Multiple-Choice Secretary Problems

Once the interviewer is allowed to accept more than one applicant, many interesting problems arise.

Sakaguchi (1978) introduces a problem with k choices with the aim to maximize the probability that any of them is the best. We define state *(r, s)* to mean that the *r*th applicant has been and the interviewer still has s choices to make. If the interviewer accepts the candidate, the transition probabilities from state *(r, s)* to *(j, s-1)* and from *(r, s)* to state $(\infty, s-1)$ are *r/j(j-1)* and *r/n* respectively. If the interviewer rejects the candidate, the transition probabilities from state *(r, s)* to *(j, s)* and from *(r, s)* to state (∞, s) are *r/j(j-1)* and *r/n* respectively. The dynamic programming equation is:

$$V(r,s) = max\left[\frac{r}{n} + \sum_{j=r+1}^{n} \frac{r}{j(j-1)} V(j,s-1), \sum_{j=r+1}^{n} \frac{r}{j(j-1)} V(j,s)\right], \tag{1.12}$$

and the one-step-policy can be evaluated by considering s=1,2,... sequentially. This determines a set of numbers $k_k^* \le k_{k-1}^* \le \ldots \le k_1^*$ such that the interviewer makes his *(k-s+1)*th choice at the first candidate to appear after item $k_s^* - 1$.

To tackle the problem in which the objective is to minimize the sum of actual ranks of the k choices, Henke (1970) shows that the *r*th applicant should be accepted, when j items have already been accepted, only if its relative rank is such that $s < s_{r,j}^*$ and Henke (1970) also gives a system of recurrence equations that determine these critical values.

Nikolaev (1977) and Tamaki (1979) solve the problem of selecting the best and second best applicants. The optimal rule says choose the first two candidates to appear after the first $r_1^* - 1$ applicants or, as second choice, the first applicant with apparent rank 2 to appear after the first $r_2^* - 1$ applicants. Sakaguchi (1979) generalizes slightly by supposing that each applicant has probability q<1 of being available if chosen. The form of the optimal policy remains unchanged, but now $r_1^* / n \to \theta$, and $r_2^* / n \to \phi$. As $n \to \infty$, the probability of winning tends to

$$\frac{\theta}{2-q}\left(2\phi^{2-q} - q\theta^{2-q}\right). \tag{1.13}$$

1.2 Problem Motivation and Description

The secretary problem and its variations have been extensively investigated. For reference, see Freeman (1983) and Ferguson (1989) for reviews. For any given problem the aim is usually to find the optimal selection strategy and relevant properties that maximize the probability of a 'win'--- that is selecting a good applicant. One of the most important feature of the classical secretary problem and its extensions is that the entire process stops at the moment we have selected the required number of applicants. In other words, there is no opportunity for the interviewer to change the selected secretaries.

However, in reality jobs are not always guaranteed and it is perfectly legitimate that managers may change their secretaries. Suppose that you need to hire a secretary, and you decide to use an employment agency which will send you one applicant per day. Then it is conceivable that you may first quickly hire a mediocre secretary to prevent a backlog of routine administrative tasks while continuing the search for a better qualified secretary.

Since a better secretary is expected to be more efficient and have a wider skill set, it may be tempting to insist on having the best possible person for the job. However, in many situations there is a significant cost involved in the switching process. To hire a new secretary is costly, since you must dismiss your current secretary and pay a large fee to the employment agency. Thus before you pay the resulting price for the replacement of a secretary, you have to estimate what that price will be. You clearly have to balance the cost of staying with an incompetent secretary for too long, against for the cost of finding a replacement. The problem becomes more complicated if you are going to hire several secretaries at a time.

In accordance with the experiment, we consider in this thesis the following variant of the multiple secretary problem. There are 100 applicants for 7 secretary positions; the quality of the applicants is uniformly distributed from 1 to 100. They are interviewed sequentially in a random order with each order being equally likely. Once rejected, an applicant cannot be recalled. The interviewer can employ at most 7 secretaries at a time, and he can replace any of them at a significant amount of switch cost. Probabilistic reasoning and dynamic programming simulation is used to determine the optimal strategy to balance the tradeoff between secretaries' competency and costs of hiring.

Possible practical applications would include:

(a) Housing lease --- The landlord has k apartments for lease, and there are n potential tenants coming to inspect the apartments one by one. Each of them would like to offer a different amount of rent. There is a cost to the landlord if he chooses to change his tenants.

(b) Development of new products --- There are n proposals for a new product coming sequentially. The company is able to produce at most k different kinds of products at a time. The management team makes the detailed productivity and marketability study of one proposal each month. It is costly to replace any existing production line by a new one.

1.3 Thesis Objectives and Organization

The main objective of the thesis is to utilize probabilistic reasoning and simulation methods to explore the optimal selection rule of the multiple secretary problem with switch costs. The problem has been presented to respondents in an experiment and the optimal selection rule developed in this work will be the benchmark against which the respondents' intuitive selection decisions are measured.

In chapter 2, we examine the effect of interview cost and opportunity cost on the optimal selection strategy in the single secretary problem with no recall. To allow a more realistic formulation of the classical secretary problem, we consider costs associated with the selection procedure. The stopping rule is found which minimizes the total cost function.

In chapter 3, the single secretary problem with switch cost is presented and its optimal selection rule is examined. This chapter starts with the infinite horizon problem, which is a simplified version of the following finite horizon problem: A constant replacement cost is incurred each time we replace a previously selected secretary with a new one. A mathematic model is created to solve the problem analytically.

We consider in chapter 4 a problem which allows us to possess more than one secretary at a time. Dynamic programming is used to solve the optimal selection rule for this problem as an extension of multiple-choice duration algorithm. Lastly, experimental results are presented and discussed.

In chapter 5, we outline possible future work directions and summarize the findings of this thesis.

Chapter 2. Generalized Secretary Problem with Costs

Before we embark on the discussion about the secretary problem with switch costs, let us first look at cost issues in the generalized secretary problem. To allow a more realistic formulation of the secretary problem, this chapter considers interview cost and opportunity cost associated with the selection procedure.

In the generalized secretary problem, in order to maximize the probability of choosing the best applicant, (i.e. to minimize the opportunity cost), the interviewer is required to interview an enormous number of applicants. Chow et al. (1964) suggests that this number is about n/4 and is as many as n/2 on the average. However, in a real situation where interview cost is not negligible, it would be wise to be less aggressive and be content with more modest results. Hence the tradeoff between opportunity cost and interview cost does exist along the whole interview process, and the optimal stopping rule can only be obtained by minimizing the total costs.

2.1 Notation

The following notations are used for the mathematical formulation.

Indices:

i	The position of an applicant along the timeline
a_i	The absolute rank of the ith applicant
r_i	The relative rank of the ith applicant, denoting the number of applicants from 1 to i whose absolute rank is smaller than or equal to a_i
s_i	The stopping rule, indicating that the interviewer stops on the first applicant for which $r_i \le s_i$

i_r The cutoff representation of the stopping rule, denoting the smallest i for which $r \le s_i$

Parameters:

n The total number of applicants

N_i The condition that no stopping occurs at the ith applicant or earlier.

x_i Expected total cost under the condition N_i

g_i Expected absolute rank of the selected applicant under the condition N_i

h_i Expected number of further interviews required under the condition N_i

c_g Cost of getting one lower expected rank

c_h Cost of interviewing one applicant

2.2 Problem Description

The payoff maximizing strategy for the classical secretary problem implies a utility function that takes the value 1 if the best applicant is selected and 0 otherwise. The generalized secretary problem can be obtained by replacing the classical utility function with a more realistic value of $\pi(a)$ where a is the absolute rank of the selected applicant. We assume $\pi(1) \ge \cdots \ge \pi(n)$.

When the interviewer's objective is to maximize earnings in the generalized secretary problem in which $\pi(a)$ increases linearly as $(n\text{-}a)$ increases, then the maximizing expected utility corresponds to minimizing expected rank of the accepted applicant. According to the optimal search policy given by Mucci (1973), the interviewer should interview and reject the first $i_1 - 1$ applicants, then between applicant i_1 and applicant $i_2 - 1$ she should only accept applicants with relative rank 1; between applicant i_2 and applicant $i_3 - 1$ she should accept applicants with relative ranks 1 or 2; and so on.

The expectation of the total cost is:

Total cost = Opportunity cost + Interview Cost

$$x_i = c_g(g_i - 1) + c_h h_i, \tag{2.1}$$

where x_i is what we want to minimize, and c_g, c_h are constant values. In order to get the optimal stopping rule to minimize the cost function, I will find out the expressions for g_i and h_i in the following sections respectively. The probability of the selection process ends at the *i*th applicant or earlier, the expected number of interviews, and the expected absolute rank of the selected applicant are among the major topics considered.

2.2.1 Probability of the Process ends at the *i*th Applicant or Earlier

Let the probability of the selection process ends at the i-th applicant or earlier be

$$N(i) = P(N_i). \tag{2.2}$$

It is important to note that the relative rank of *i*th applicant r_i is independent of the values of $r_1, r_2, \cdots, r_{i-1}$. Since the value of r_i is determined by the absolute ranks of the previous *i-1* applicants, r_i takes the value from 1 to *i* with equal probability. In other words, relative ranks of the first *i* applicants can appear as any one of the *i!* permutations with uniform probability *1/i!*.

Let the probability of stopping at the *i*th applicant be

$$S(i) = P\{N_{i-1}, r_i \le s_i\} = P(N_{i-1})P(r_i \le s_i) = N(i-1) \cdot \frac{s_i}{i}. \tag{2.3}$$

The probability of the process ends at the *i*th applicant or earlier is given by

$$N(i) = N(i-1) - S(i) = N(i-1) - N(i-1) \cdot \frac{s_i}{i}. \tag{2.4}$$

$N(i)$ can be derived as follows: the event N_i implies that the first applicant whose relative rank is 1 definitely appears in the first $i_1 - 1$ positions, and the second applicant whose relative rank is 1 or 2 must appear in the first $i_2 - 1$ positions except the position occupied by the applicant of relative rank 1, ..., the rth applicant with relative rank from 1 to r is in the first $i_r - 1$ positions except the positions occupied by the previous *r-1* applicants. The remaining *(i-r)* applicants can be in any position among the unoccupied *(i-s)* positions because $i \le i_{r+1}$. Hence there are

$$\prod_{k=1}^{r} (i_k - k) \cdot (i-r)! \tag{2.5}$$

permutations corresponding to N_i, and this leads to the following expression

$$N(i) = \frac{\prod_{k=1}^{r} (i_k - k) \cdot (i-r)!}{i!}. \tag{2.6}$$

2.2.2 The Expected Number of Interviews

The probability of event N_i equals to the sum of the probabilities of the processes ending at each position from *i+1* to *n*. Thus for any *i*

$$N(i) = S(i+1) + S(i+2) + \cdots + S(n). \tag{2.7}$$

The expected number of interviews until the end of process is then given by

$$h_0 = \sum_{i=1}^{n} S(i) \cdot i = 1 + \sum_{i=1}^{n-1} N(i). \tag{2.8}$$

At the ith position, the expected number of additional interviews until the end of process condition on N_i is given by

$$h_i = \frac{1}{N(i)}\sum_{k=1}^{n-i} k \cdot S(i+k) = \frac{1}{N(i)}\sum_{k=i}^{n} N(k)\,. \tag{2.9}$$

Similarly we can get

$$h_{i-1} = \frac{1}{N(i-1)}\sum_{k=i-1}^{n} N(k)\,. \tag{2.10}$$

From (2.8), (2.9) and (2.10), we can get a recurrence formula for the expected number of interviews, which is given as,

$$h_{i-1} = h_i \cdot \frac{N(i)}{N(i-1)} + 1 = h_i \cdot (1 - \frac{s_i}{i}) + 1 \tag{2.11}$$

2.2.3 The Expected Absolute Rank of the Selected Applicant

Given that there are n applicants, the probability that the ith applicant has an absolute rank of a, condition on having a relative rank of r_i is given by Lindely (1961) as,

$$P(a_i = a \mid r_i = r) = \frac{\binom{a-1}{r-1}\binom{n-a}{i-r}}{\binom{n}{i}}, \tag{2.12}$$

for $r \le a \le r+n-i$. Otherwise $P(a_i = a \mid r_i = r) = 0$.

The expected absolute rank under the condition that the relative rank of the ith applicant is r is given by

$$E\left(a_i \mid r_i = r\right) = \sum_{a=1}^{n} a \cdot P\left(a_i = a \mid r_i = r\right) \tag{2.13}$$

$$= \sum_{a=r}^{n+r-i} a \cdot \frac{\binom{a-1}{r-1}\binom{n-a}{i-r}}{\binom{n}{i}}$$

$$= \sum_{a=r}^{n+r-i} a \cdot \frac{\frac{(a-1)!}{(a-r)!(r-1)!}\binom{n-a}{i-r}}{\binom{n}{i}}$$

$$= \sum_{a=r}^{n+r-i} r \cdot \frac{\frac{a(a-1)!}{r(a-r)!(r-1)!}\binom{n-a}{i-r}}{\binom{n}{i}}$$

$$= \sum_{a=r}^{n+r-i} r \cdot \frac{\binom{a}{r}\binom{n-a}{i-r}}{\binom{n}{i}}$$

$$= \frac{r}{\binom{n}{i}} \sum_{a=r}^{n+r-i} \binom{a}{r}\binom{n-a}{i-r}$$

$$= \frac{r}{\binom{n}{i}} \cdot \binom{n+1}{i+1}$$

$$= \frac{(n+1)!}{(i+1)!(n-i)!} \cdot \frac{r(n-i)!i!}{n!}$$

$$= \frac{n+1}{i+1} r .$$

Hence the expected absolute rank can be obtained, from (2.3) and (2.13), as

$$E(a) = \sum_{i=i_1}^{n} \sum_{a=1}^{n} a \cdot P(a_i = a \mid r_i = r) S(i) \tag{2.14}$$

$$= \sum_{i=i_1}^{n} \sum_{a=1}^{n} a \cdot P(a_i = a \mid r_i = r) P(N_{i-1}) P(r \le s_i)$$

$$= \sum_{i=i_1}^{n} \sum_{a=1}^{n} a \cdot P(a_i = a \mid r_i = r) N(i-1) P(r \le s_i)$$

$$= \sum_{i=i_1}^{n} \sum_{r=1}^{s_i} \frac{n+1}{i+1} r \cdot N(i-1) \frac{1}{i}$$

$$=\sum_{i=i_1}^{n}\frac{n+1}{i+1}N(i-1)\frac{1}{i}\cdot\frac{s_i(s_i+1)}{2}.$$

Let g_i denote the expected rank condition on N_i, i.e.

$$g_i = E(a \mid N_i). \tag{2.15}$$

Using (2.4) and (2.15), we can get the recurrence formula for the expected absolute rank condition on N_i.

$$g_{i-1} = \frac{1}{N(i-1)}\sum_{k=i}^{n}\frac{n+1}{k+1}N(k-1)\frac{1}{k}\cdot\frac{s_k(s_k+1)}{2} \tag{2.16}$$

$$=\frac{n+1}{i+1}\cdot\frac{1}{i}\cdot\frac{s_i(s_i+1)}{2}+g_i\frac{N(i)}{N(i-1)}$$

$$=\frac{n+1}{i+1}\cdot\frac{1}{i}\cdot\frac{s_i(s_i+1)}{2}+g_i\left(1-\frac{s_i}{i}\right)$$

2.3 Optimal Stopping Rule

We have derived the expressions for expected number of interviews and expected absolute rank condition on N_i, and thus we have

$$x_i = c_g\left(g_i - 1\right) + c_h h_i \tag{2.17}$$

Using (2.11) and (2.17), we can get the recurrence formula for total cost x_i under condition N_i as

$$x_{i-1} = c_g\left(g_{i-1}-1\right)+c_h h_{i-1} \tag{2.18}$$

$$=c_g\left(\frac{n+1}{i+1}\cdot\frac{1}{i}\cdot\frac{s_i(s_i+1)}{2}+g_i\left(1-\frac{s_i}{i}\right)-1\right)+c_h\left(h_i\cdot\left(1-\frac{s_i}{i}\right)+1\right)$$

$$=c_g\frac{n+1}{i+1}\cdot\frac{1}{i}\cdot\frac{s_i(s_i+1)}{2}+c_g g_i\left(1-\frac{s_i}{i}\right)-c_g\left(1-\frac{s_i}{i}\right)+c_h h_i\cdot\left(1-\frac{s_i}{i}\right)+c_h-c_g\frac{s_i}{i}$$

$$= c_g \frac{n+1}{i+1} \cdot \frac{1}{i} \cdot \frac{s_i(s_i+1)}{2} + c_h - c_g \frac{s_i}{i} + x_i \left(1 - \frac{s_i}{i}\right)$$

$$= x_i + c_h + \frac{1}{i} \sum_{j=1}^{s_i} \left(c_g \frac{n+1}{i+1} j - c_g - x_i \right).$$

It becomes obvious that s_i is to be chosen as the largest integer j which satisfies

$$c_g \frac{n+1}{i+1} j \le c_g + x_i, \tag{2.19}$$

namely

$$s_i = \left\lfloor \frac{i+1}{n+1} \left(1 + \frac{x_i}{c_g} \right) \right\rfloor. \tag{2.20}$$

Since $g_{n-1} = \frac{n+1}{2}$ and $h_{n-1} = 1$, we get

$$x_{n-1} = c_g \left(g_{n-1} - 1 \right) + c_h h_{n-1} = c_g \frac{n-1}{2} + c_h. \tag{2.21}$$

Starting from x_{n-1}, we can determine the values of the optimal stopping rule s_i and the optimized expected total cost x_i by using (2.18) and (2.20) respectively.

The computations discussed above can be done by using computer simulation. The following graphs (Figure 1, 2) presents the results for an example where the total number of applicants is $n=100$, the cost of getting one lower expected rank is $c_g = 2$, and the cost of interviewing one applicant c_h are 0 and 3. In Figure 1, it is interesting to note that one should keep observing about 30% of the population without taking any action. After that the interviewer takes an applicant only if it turns out to be the best among all those observed so far. At the end of this phase, about 50% of the population has been observed.

Figure 2 presents different curves due to the existence of interview costs. The waiting period reduces significantly and the interviewer has to make decisions as early as at $i=4$. In other words, the stopping rule gets mild much sooner. The optimal expected total cost decreases as time increases from 1 to 82 but increases for time greater than 82. In this case, the stopping rule is much more relaxed when compared to that in the previous case. This is because the interviewer must pay an interview cost for each observation of a new applicant, and he must strike a balance between the costs of interviewing and secretaries' competency.

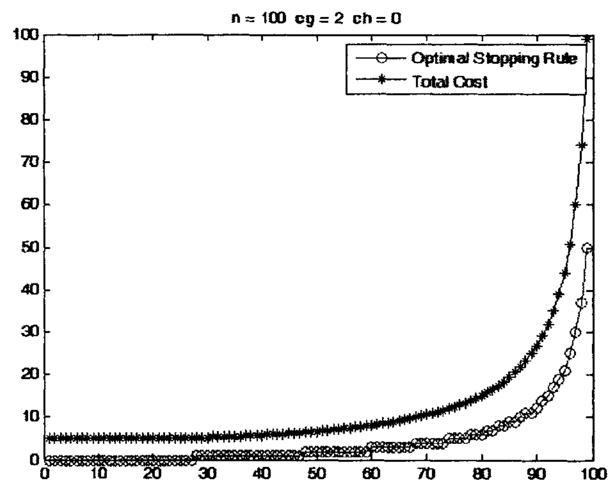

Figure 1. Generalized Secretary Problem with Cost Coefficients $c_g = 2$ and $c_h = 0$

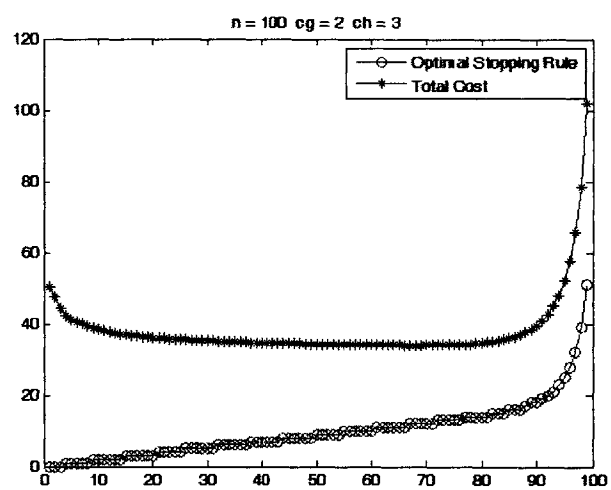

Figure 2. Generalized Secretary Problem with Cost Coefficients $c_g = 2$ and $c_h = 3$

Chapter 3. Single Secretary Problem with Switch Costs

In this chapter we consider the following variant of the secretary problem. There are n of applicants for a single position and n is known in advance. The applicants are interviewed sequentially in a random order with each interview taking one time unit. The quality of each applicant is uniformly distributed. The hired secretary can be dismissed at any time. The number of replacements is not limited, but there is a switch cost involved in this substitution process, consequently a significant increased cost could be incurred if redundant changes are made.

Another key difference between the generalized secretary problem and the single secretary problem with switch costs is the way we define the payoff. Since the job position could be occupied by different secretaries along the interview process, the payoff is no longer defined as the quality of one particular secretary, Instead it is now defined as the weighted average quality of all the selected secretaries. In this problem, the payoff is stated as the sum of the qualities of secretaries at each time unit less total switch costs incurred.

$$Payoff = \left(\sum_{i=1}^{n} quality\ of\ secretary\ at\ time\ i \right) - \left(switch\ cost \times No.\ of\ switches \right) \quad (3.1)$$

Because the interview cost is almost negligible compared to the relatively large switch cost, the interviewer will see all the applicants in this case. Taking the extreme case where the switch cost is zero, we can see that the interviewer would definitely insist on having the best possible person for the job at all times; at the end of each interview, he will replace the current secretary with the current applicant if the current applicant is better qualified. However, in reality where switch cost is significant, a certain degree of quality "superiority" is needed before a switch become worthwhile. This amount of quality superiority required is described as the selection criteria, which is one of the key issues we explore in this problem.

We first consider the infinite horizon problem where the number of available applicants are infinite. In Section 3.2, we consider the problem where the time horizon is finite, which is more complicated because the interviewer has to deal with different time constraints along the selection process.

3.1 The Infinite Horizon Problem

In this problem, an infinite number of applicants whose qualities are i.i.d. are interviewed sequentially. We received a payoff as long as we are employing a secretary, and the payoff is defined as the quality of the secretary multiplied by the duration of time we are employing this secretary. The total payoff is the sum of payoffs received from all the secretaries whom we had hired or currently exploying, less the total switching costs. The objective of this chapter is to find the optimal rule to maximize the total payoff.

3.1.1 Mathematical Model

We assume that $X_1, X_2, \cdots$ are independent and identically distributed (i.i.d.) random variables, uniformly distributed over [0, 1], where X_j denotes the quality of the secretary at the jth stage. Let T_i denote the length of time we are in possession of the ith secretary. Given that switch cost is a constant d, the payoff received from the ith secretary is:

$$P_i = \begin{cases} X_i \cdot T_i, & i = 1 \\ X_i \cdot T_i - d, & i \geq 2 \end{cases}. \tag{3.2}$$

If n secretaries are to be hired during the entire selection process, then the total payoffs can be written as

$$\sum_{i=1}^{n} P_i = X_1 \cdot T_1 + \sum_{i=2}^{n} X_i \cdot T_i - d \ . \tag{3.3}$$

3.1.2 Problem Analysis

We should note that the new secretary to be hired must have a higher quality than the one we currently have. However, it is neither necessary nor optimal to accept every applicant whose quality is higher than that of the current secretary. This is because in the situation where there is a switch cost involved, there would be a reduced loss in making redundant replacements. If the quality of the current hired secretary is X_i, we might not make the replacement until we meet an applicant with quality X_{i+1}, which is sufficiently higher than X_i. In other words, the value of X_{i+1} is always greater than that of X_i for every i. Therefore, we can define the optimal policy as follows:

Conditional on the quality of the current secretary being X_i, a replacement will be made on the first applicant whose quality is $X_{i+1} \geq X_i + \varepsilon_i$, where $X_0 = 0$, and ε_i is defined as ith optimal policy coefficient.

The optimal strategy can be summarized as follows.

Theorem 1. *Let X_i denote the quality of ith secretary, and ε_i denote the ith optimal policy coefficient. Then the expected length of time we are in possession of this secretary is:*

$$E[T_i] = \frac{1}{1 - X_i - \varepsilon_i} \tag{3.4}$$

Proof. The optimal rule indicates that a replacement will be made on the first applicant whose quality is $X_{i+1} \geq X_i + \varepsilon_i$. Thus, each time when we meet a new applicant the probability of acceptance is:

$$P_A = (1 - X_i - \varepsilon_i); \tag{3.5}$$

and the probability of rejection is

$$P_B = 1 - P_A = (X_i + \varepsilon_i). \tag{3.6}$$

Therefore, given the quality of the current secretary and the corresponding optimal policy coefficient, we can get the expected duration during which we are in possession of this secretary:

$$\begin{aligned} E[T_i] &= 1P_A + 2P_AP_B + 3P_AP_B^{\,2} + 4P_AP_B^{\,3} + \ldots + nP_AP_B^{\,n-1} \qquad (3.7) \\ &= 1\cdot(1 - X_i - \varepsilon_i) + 2\cdot(1 - X_i - \varepsilon_i)(X_i + \varepsilon_i) + \cdots + n\cdot(1 - X_i - \varepsilon_i)(X_i + \varepsilon_i)^{n-1} \\ &= 1 - (X_i + \varepsilon_i) + 2(X_i + \varepsilon_i) - 2(X_i + \varepsilon_i)^2 + \cdots + n(X_i + \varepsilon_i)^{n-1} - n(X_i + \varepsilon_i)^n \\ &= \frac{1}{1 - X_i - \varepsilon_i} \end{aligned}$$

Theorem 2. *Let A denote a set of elements uniformly distributed on the interval [a, 1], and let B denote a set of elements uniformly distributed on the interval [b, 1]. If we can map A onto B and the transformation is a one-to-one correspondence, then we can map* $a + \varepsilon_a$ *in A onto* $b + \varepsilon_b$ *in B, where*

$$\varepsilon_b = \varepsilon_a \frac{1 - b}{1 - a} \tag{3.8}$$

Proof. The linear mapping from A onto B can be expressed in the following function:

$$f(x) = \frac{x - a}{1 - a}(1 - b) + b \tag{3.9}$$

Obviously it is a bijective function from A to B with the property that, for every $n \in B$, there is exactly one m in A such that $f(m) = n$. Therefore,

$$f(a+\varepsilon_a) = \frac{\varepsilon_a}{1-a}(1-b)+b \tag{3.10}$$
$$= b + \varepsilon_a \frac{1-b}{1-a}$$
$$= b + \varepsilon_b$$
$$\Rightarrow \varepsilon_b = \varepsilon_a \frac{1-b}{1-a}.$$

In a special case when $a=0$, $\varepsilon_b = \varepsilon_a \frac{1-b}{1-a} = \varepsilon_a (1-b)$

Theorem 3. *If the optimal policy dictates that the first secretary will be selected on an applicant whose quality is $X_1 \geq \varepsilon_0$, the ith optimal policy coefficient can be written as:*

$$\varepsilon_i = (1-X_i)\cdot\varepsilon_0 \tag{3.11}$$

Proof. Suppose we know the optimal policy coefficients $\varepsilon = (\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_n)$ to select secretaries whose qualities are uniformly distributed over [0, 1]. Then the expected optimal payoffs of using the given optimal policy are:

$$E[P] = \sum_{i=1}^{n} E[P_i] = E[X_1 \cdot E[T_1]] + \sum_{i=2}^{n} E[X_i \cdot E[T_i]] - d, \tag{3.12}$$

where n denotes the total number of secretaries hired in the whole process, and X_i is strictly selected based on the optimal policy. The probability of acceptance is:

$$P(X_i) = \begin{cases} 1/(1-X_{i-1}-\varepsilon_{i-1}), & (X_{i-1}+\varepsilon_{i-1}) < X_i \leq 1 \\ 0, & otherwise \end{cases}; \tag{3.13}$$

and we define that we have a "dummy secretary" $X_0 = 0$ at the beginning of the process.

Suppose we rescale the quality distribution of the secretaries into [b, 1], where $b > 0$. Then according to Theorem 2, the value of X_i will be consequently rescaled to $X_i' = X_i(1-b)+b$ and the value of ε_i becomes $\varepsilon_i' = \varepsilon_i(1-b)$.

According to Theorem 3, the expected length of time we are in possession of this particular secretary becomes:

$$E[T_i'] = \frac{1}{1 - X_i' - \varepsilon_i'} = \frac{1}{(1-b) - X_i(1-b) - \varepsilon_i(1-b)} = \frac{E[T_i]}{1-b}. \tag{3.14}$$

Suppose after rescaling the quality distribution interval, the selection policy coefficients are $\varepsilon' = (\varepsilon_0', \varepsilon_1', \cdots, \varepsilon_n')$, then

$$\begin{aligned}
&(X_{i-1} + \varepsilon_{i-1}) < X_i \le 1 &(3.15)\\
&\Rightarrow (1-b)(X_{i-1} + \varepsilon_{i-1}) < (1-b) X_i \le (1-b)\\
&\Rightarrow (1-b) X_{i-1} + (1-b)\varepsilon_{i-1} < (1-b) X_i \le 1-b\\
&\Rightarrow (1-b) X_{i-1} + b + (1-b)\varepsilon_{i-1} < (1-b) X_i + b \le 1\\
&\Rightarrow X_{i-1}' + \varepsilon_{i-1}' < X_i' < 1.
\end{aligned}$$

Though the optimality of this selection policy is yet to be determined, we can get the expected total payoff based on the policy coefficients $\varepsilon' = (\varepsilon_0', \varepsilon_1', \cdots, \varepsilon_n')$:

$$E[P'] = \sum_{i=1}^{n} E[P_i'] \tag{3.16}$$

$$= E[X_1' \cdot E[T_1']] + \sum_{i=2}^{n} E[X_i' \cdot E[T_i']] - d$$

$$= E\left[((1-b)X_1 + b) \cdot \frac{E[T_1]}{1-b}\right] + \sum_{i=2}^{n} E\left[((1-b)X_i + b) \cdot \frac{E[T_i]}{1-b}\right] - d$$

$$= E\left[X_1 \cdot E[T_1] + \frac{b \cdot E[T_1]}{1-b}\right] + \sum_{i=2}^{n} E\left[X_i \cdot E[T_i] + \frac{b \cdot E[T_i]}{1-b}\right] - d.$$

In this problem, we are only interested in the incremental total payoffs. The basic payoff, which can be obtained by keeping the original secretary with quality b without any replacement along the whole process, should be subtracted from the total payoffs. The incremental payoff can be calculated as:

$$E[P'] - b \cdot E[T] \tag{3.17}$$

$$= \sum_{i=1}^{n} E[P_i'] - \sum_{i=1}^{n} \frac{b \cdot E[T_i]}{1-b}$$

$$= E\left[X_1 \cdot E[T_1] + \frac{b \cdot E[T_1]}{1-b}\right] + \sum_{i=2}^{n} E\left[X_i \cdot E[T_i] + \frac{b \cdot E[T_i]}{1-b}\right] - d - \sum_{i=1}^{n} \frac{b \cdot E[T_i]}{1-b}$$

$$= E\left[X_1 \cdot E[T_1]\right] + \sum_{i=2}^{n} E\left[X_i \cdot E[T_i]\right] - d$$

$$= E[P],$$

By following the selection policy $\varepsilon' = (\varepsilon_0', \varepsilon_1', \cdots, \varepsilon_n')$, we can get the incremental total payoffs which is the same as the total payoffs with uniform distribution on [0, 1].

Suppose the *optimal selection policy coefficients* to yield the maximum additional total payoffs with uniform distribution on [b, 1] are $\tilde{\varepsilon} = (\tilde{\varepsilon}_0, \tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_n)$. By following this optimal selection policy, the optimal incremental total payoffs can be obtained by:

$$E\left[\tilde{P}'\right] - b \cdot E\left[\tilde{T}\right] = E\left[\tilde{X}_1 \cdot E\left[\tilde{T}_1\right]\right] + \sum_{i=2}^{n} E\left[\tilde{X}_i \cdot E\left[\tilde{T}_i\right]\right] - d, \tag{3.18}$$

where

$$\tilde{X}_{i-1} + \tilde{\varepsilon}_{i-1} < \tilde{X}_i < 1.$$

Obviously when $b=0$, the optimal incremental total payoffs are supposed to be equivalent to the optimal total payoffs with uniform distribution on [0, 1]:

$$E\left[\tilde{P}'\right] - 0 \cdot E\left[\tilde{T}\right] = E[P'] \tag{3.19}$$

$$\Rightarrow E\left[\tilde{X}_1 \cdot E\left[\tilde{T}_1\right]\right] + \sum_{i=2}^{n} E\left[\tilde{X}_i \cdot E\left[\tilde{T}_i\right]\right] - d = E\left[X_1 \cdot E[T_1]\right] + \sum_{i=2}^{n} E\left[X_i \cdot E[T_i]\right] - d.$$

We can see that the same amount of payoffs can be obtained by using either selection policy $\tilde{\varepsilon} = (\tilde{\varepsilon}_0, \tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_n)$ or $\varepsilon' = (\varepsilon_0', \varepsilon_1', \cdots, \varepsilon_n')$. Hence we can conclude that $\varepsilon' = (\varepsilon_0', \varepsilon_1', \cdots, \varepsilon_n')$ are the optimal selection policy coefficients, which if used would yield the maximum incremental total payoffs.

In the infinite time horizon problem, the selection process is memoryless because at any stage the remaining time is infinite. This suggests that every point along the time horizon is a fresh start but with a distinct starting point 'b'. Thus we can always follow the optimal policy to get the optimal incremental total payoffs. The ith optimal policy coefficient is therefore

$$\varepsilon_i = (1 - X_i) \cdot \varepsilon_0 \tag{3.20}$$

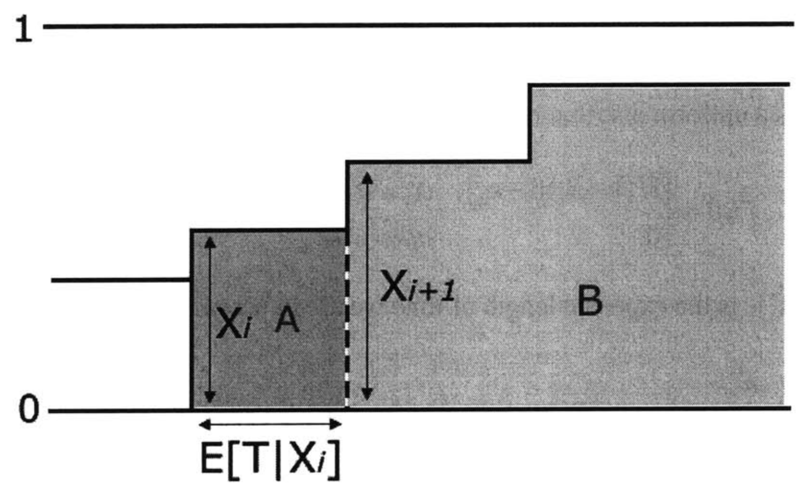

Figure 3. Payoffs after Selecting the Secretary with Quality X_i

In the optimal selection process, let $P(X_i)$ denote the total payoffs after selecting the secretary with quality X_i. The value of $P(X_i)$ can be divided into two parts as illustrated in Figure 3. The first part deals with the payoff to be earned as long as the current secretary remains employed, which is labeled as region A. This portion of payoff is mathematically defined as the secretary's quality X_i times the expected time of possession of this particular secretary minus the switch cost. The second part, which is labeled as region B, is the total payoffs to be earned $P\left(X_{i+1}\right)$, where X_{i+1} is the quality of the next selected secretary. Thus,

$$P(X_i) = X_i \times E[T \mid X_i] - d + P(X_{i+1}). \tag{3.21}$$

In this equation, X_{i+1} depends on X_i and ε_i, and ε_i is determined by X_i and ε_0. Assuming that the value of ε_0 is known, we can find the value of X_{i+1} which is completely dependent on X_i:

$$X_i + (1 - X_i) \cdot \varepsilon_0 \leq X_{i+1} \leq 1. \tag{3.22}$$

Let $P(X_i)$ denote the total payoffs if the quality of the current secretary is X_i, then

$$E[P(X_i)] = X_i \times E[T \mid X_i] - d + \int_{X_i + (1-X_i)\varepsilon_0}^{1} E[P(x)] f(x) dx. \tag{3.23}$$

where *f(x)* is a uniform distribution on $[X_i + (1 - X_i) \cdot \varepsilon_0, 1]$:

$$f(x) = \begin{cases} 1/(1 - X_i)(1 - \varepsilon_0), & X_i + (1 - X_i) \cdot \varepsilon_0 < x \leq 1 \\ 0, & \textit{otherwise} \end{cases}; \tag{3.24}$$

and $E[T \mid X_i]$ is the expected length of time we are in possession of the secretary with quality X_i:

$$E[T \mid X_i] = \frac{1}{1 - X_i - (1 - X_i)\varepsilon_0} = \frac{1}{(1 - X_i)(1 - \varepsilon_0)}. \tag{3.25}$$

Let $y(X_i)$ denote the expected total payoffs $E[P(X_i)]$:

$$\begin{aligned} y(X_i) &= X_i \cdot \frac{1}{(1 - X_i)(1 - \varepsilon_0)} - d + \int_{X_i + (1-X_i)\varepsilon_0}^{1} y(x) \frac{1}{(1 - X_i)(1 - \varepsilon_0)} dx \qquad (3.26) \\ &= X_i \cdot \frac{1}{(1 - X_i)(1 - \varepsilon_0)} - d + \frac{1}{(1 - X_i)(1 - \varepsilon_0)} \int_{X_i + (1-X_i)\varepsilon_0}^{1} y(x) dx. \end{aligned}$$

Because we have no secretary at the beginning of the selection process, the expected total payoffs with uniform distribution on [0, 1] is $y(0)$. Thus the objective of the problem is to find the optimal policy coefficient ε_0^* which maximizes $y(0)$. By knowing the optimal value of ε_0, the values of optimal policy coefficients $\varepsilon = (\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_n)$ can be easily obtained by using the formula of $\varepsilon_i = (1 - X_i) \cdot \varepsilon_0$.

3.1.3 The Alternative Methodology

We can look at this problem from another analytical angle. Instead of maximizing the expected payoffs, we can find the optimal policy coefficient ε_0^* by minimizing the expected total costs. The expected total costs consist of both opportunity costs and switch costs. Opportunity costs are defined as the additional payoffs which could be received if the quality of the encountered secretary is 1.

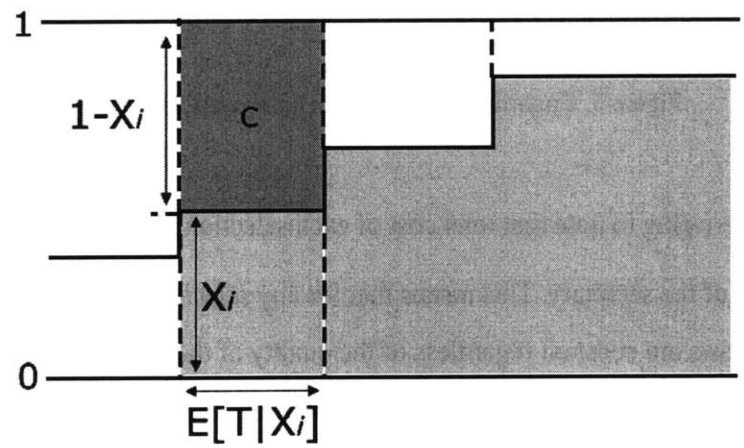

Figure 4. Alternative Method to Determine the Optimal Policy Coefficients

Let us denote the expected total costs of each selection as $L(X_i)$ given that the selected secretary is X_i. As illustrated in Figure 4, the opportunity cost for each selected secretary is equal to the size of region C. Since the switch cost is a constant *d*, the expected total cost $L(X_i)$ for secretary with quality X_i can be expressed as:

$$L(X_i) = Opportunity\ Cost + Switch\ Cost \tag{3.27}$$

$$= (1 - X_i) \cdot E[T \mid X_i] + d$$

$$= (1 - X_i) \cdot \frac{1}{(1 - X_i)(1 - \varepsilon_0)} + d$$

$$= \frac{1}{(1 - \varepsilon_0)} + d$$

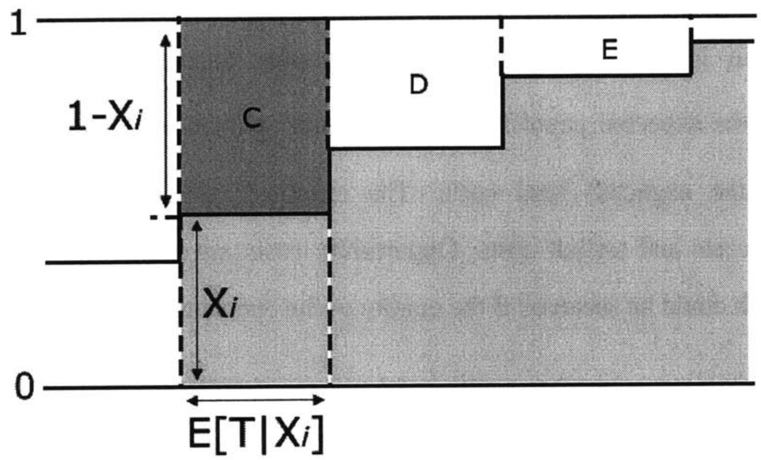

Figure 5. Opportunity Cost for Each Selected Secretary

It is interesting to note that total cost of each selection $L(X_i)$ is independent of the quality of the secretary. This means that for any selected secretary, the opportunity costs are constant regardless of the quality of the secretary. From a graphic point of view, the sizes of region C, D and E in Figure 5 are exactly the same. Thus instead of maximizing the value of $y(0)$, we can try to minimize the total loss in order to get the optimal value of ε_0. Let $N(X_i)$ denote the expected number of replacements to be made when the quality of the selected secretary is X_i. The total loss is defined by

$$L \cdot N(X_i) \tag{3.28}$$

where

$$N(X_i) = 1 + \int_{X_i+(1-X_i)\varepsilon_0}^{1} N(x) f(x)\, dx \tag{3.29}$$

$$= 1 + \frac{1}{(1-X_i)(1-\varepsilon_0)} \int_{X_i+(1-X_i)\varepsilon_0}^{1} N(x)\, dx$$

The objective of the problem is to find the optimal policy coefficient ε_0^* which minimizes $L \cdot N(X_i)$.

3.1.4 Simulation Results and Analysis

We will solve integral equation (3.29) by using a Laplace Transform. Starting with

$$y(x) = 1 + \frac{1}{(1-x)(1-\varepsilon)} \int_{(1-\varepsilon)x+\varepsilon}^{1} y(t)\, dt \tag{3.30}$$

$$\Rightarrow y(x) = 1 - \frac{1}{(1-x)(1-\varepsilon)} \int_{1}^{(1-\varepsilon)x+\varepsilon} y(t)\, dt$$

$$\Rightarrow y(x) = f(x) + g(x) \int_{a}^{bx+c} y(t)\, dt,$$

where $f(x) = 1$, $g(x) = -\frac{1}{(1-x)(1-\varepsilon)}$, $a = 1$, $b = 1-\varepsilon$ and $c = \varepsilon$.

We can apply Laplace Transform to both sides of equation (3.30), and put $x = s + \frac{c}{1-b} = s+1$, $t = u + \frac{c}{1-b} = u+1$, the equation yields

$$Y(s) = F(s) + G(s) \int_{a-\frac{c}{1-b}}^{bs} Y(u)\, du \tag{3.31}$$

$$\Rightarrow Y(s) = 1 + \frac{1}{s(1-\varepsilon)} \int_{0}^{bs} Y(u)\, du$$

Let $Z(s) = \int_0^s Y(u)\, du$, it is easy to see that Z(0)=0. Thus we can get

$$Z'(s) = 1 + \frac{1}{(1-\varepsilon)s} Z((1-\varepsilon)s). \tag{3.32}$$

However the rescaling factor $(1-\varepsilon)$ makes the Laplace equation (3.32) hard to solve analytically.

We can employ recursion to solve integral equation (3.29). In accordance with the experiment, we rescale the secretary quality distribution to [0, 100], and set switch cost as 500. The following is the corresponding pseudo-code to calculate the expected payoffs for one particular value of ε:

```
function (eps, x)
    lowlimit = (1-ε)x+ε
    counter = 0
    for i = lowlimit to 100 do
        sum = sum + function (eps, i)
        counter ++
    end for
    return 1+sum/counter
end function
```

The optimal value of ε_0 is the one that minimizes total opportunity cost $y(x)$. Figure 6, which compares the points showing (3.29) with the results for different values of policy coefficients ε_0, indicates that the optimal policy coefficient to achieve minimum total costs is $\varepsilon_0^* = 67$. The total cost decreases as ε_0 increases from 1 to 67 but increases as ε_0 goes beyond 67. This suggests that quality of the first secretary we are supposed to select along the interview process should be greater than 67. The optimal policy coefficients for the later stages can be calculated as:

$$\varepsilon_i = \left(1-\frac{X_i}{100}\right)\cdot\varepsilon_0^* = \left(1-\frac{X_i}{100}\right)\cdot 67 \tag{3.33}$$

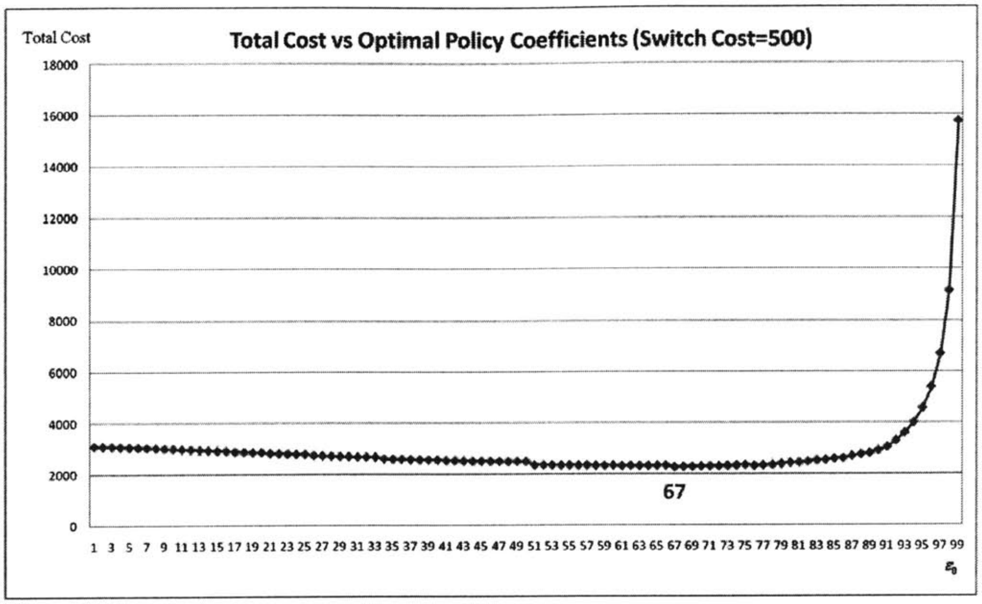

Figure 6. Total Cost vs. Optimal Policy Coefficients (Switch Cost = 500)

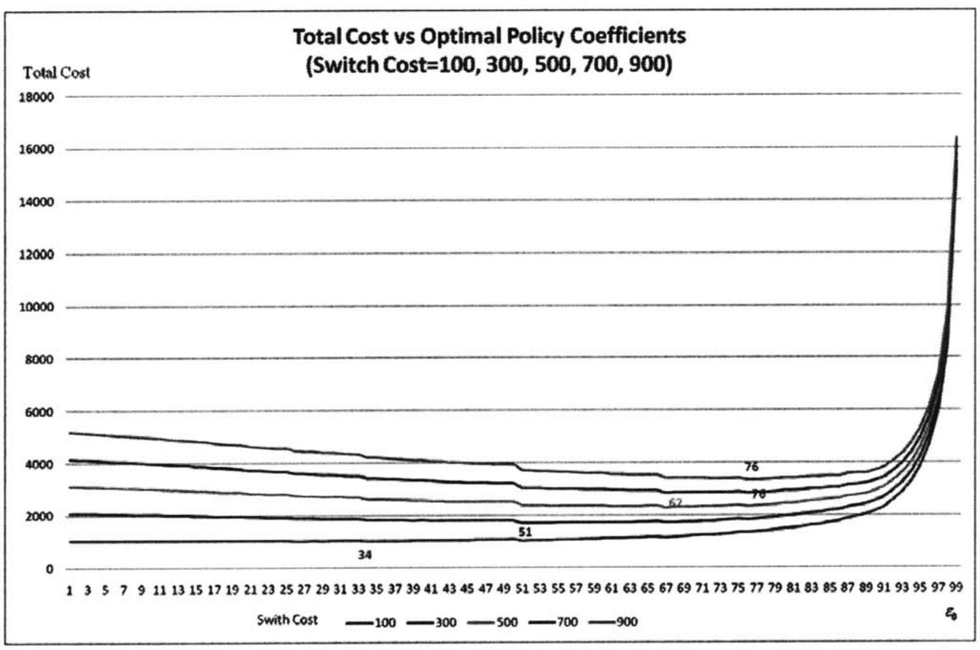

Figure 7. Total Cost vs. Optimal Policy Coefficients (Switch Cost = 100, 300, 500, 700, 900)

Figure 7 compares how the total cost curve varies across different values of switch costs *d*. The total costs associated with a greater switch cost are generally higher than those associated with a smaller one, and the curves converge as ε_0 approaches 100. Table 2 shows the optimal policy coefficients with different switch costs. It can be seen that the optimal policy coefficients increase with the value of switch costs and converge to 76.

Switch Cost	Optimal ε_0^*	Total Cost
100	34	1009.46
300	51	1692.48
500	67	2275.62
700	76	2837.82
900	76	3346.08
1100	76	3854.35

Table 2. Optimal Policy Coefficients Associated with Different Values of Switch Costs

3.2 The Finite Horizon Problem

The finite horizon problem is essentially the same as the infinite horizon case except that the number of applicants to be seen is finite. In this thesis, the number of applicants is set as $n=100$ in line with the situation in the experiment. As we mentioned earlier, the selection process in the infinite time horizon problem is memoryless because at any stage the remaining time is always infinite in the infinite horizon case. This suggests that the problems we encounter at each point along the process are basically the same.

However, in the finite horizon case, different time constraints may yield different selection decisions. At the beginning of the interview process, the selection rule might be similar to those in the infinite horizon problem because the interviewer has enough time to achieve significant incremental payoffs. When the interview process approaches its end, the interviewer may hesitate to change secretary because the additional payoffs to be earned from the new secretaries might not enough to recoup the relatively large switch costs incurred.

3.2.1 Single Choice Duration Problem

Ferguson (1991) introduces a sequential observation and selection problem called the duration problem, which is related to the single secretary problem with switch cost we consider here. The distinguish feature of the duration problem is that the payoff to the interviewer is the length of time he is in possession of a relatively best object. Thus, he will only select a relatively best applicant, receiving a payoff of one as he does so and an additional one for each new observation as long as the selected applicant stays relatively best.

We assume that X_1, X_2, ... are i.i.d. random variables, uniformly distributed on [0, 1], where X_n denotes the quality of the applicant at the *n*th stage from the end. Let *w(x, n)* denote the expected payoff given that the *n*th applicant from the last is a relatively best applicant of quality $X_n = x$ and we select it.

$$w(x,n) = 1 + x + x^2 + \cdots x^{n-1} = \frac{1-x^n}{1-x} \tag{3.34}$$

We can use backward induction to find the optimal rule. Let *v(x, n)* denote the optimal expected return when there are *n* applicants yet to be observed and the present maximum of past observation is *x*. *v(x, n)* can be defined as

$$v(x,n) = xv(x,n-1) + \int_x^1 max\{w(t,n), v(t,n-1)\}dt \tag{3.35}$$

with initial condition, *v(x, n)=0.*

Let *u(x, n)* denote the expected payoff when we skip a relatively best applicant and select the next relatively best applicant if any. We can express *u(x, n)* as

$$u(x,n) = \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t,n-k)\,dt \tag{3.36}$$

$$=\sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} \frac{1-x^j}{j}.$$

Obviously we should select the secretary if $w(x,n) \geq u(x,n)$. Thus in the full information duration problem, it is optimal to select a relatively best secretary of quality $x \geq x_n$ at n stages from the end, where x_n is the unique root of the equation,

$$\sum_{k=1}^{n} x^{k-1} = \sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} \frac{1-x^j}{j} \tag{3.37}$$

Ferguson (1991) uses numerical methods to solve this equation and gave the approximate root of the equation to be $x \approx 1 - \frac{2.12}{n}$.

Though this duration problem is similar to the single secretary problem with switch costs in that the payoff is related to the length of time the interviewer is in possession of a secretary. However unlike the single secretary problem, the payoff defined in the duration problem is independent of the specific values of the selected secretaries, and is only based on one particular secretary.

3.2.3 Numerical Analysis

In the single secretary problem with switch costs, we are allowed to employ only one applicant at any time and a constant cost *d* is incurred when replacement takes place. Imagine a situation where we employ a secretary and a new candidate has appears. We need to decide if we want to accept the new candidate and dismiss the previous secretary at a cost of *d* or reject the new candidate and continue employing the current secretary. For simplicity we do not consider interview cost in this problem.

We consider this problem as a Markov decision process model. Decision of either selection or rejection takes place only when a candidate appears. We describe the state of the process as *(x, y, n)*, $1 \leq x \leq 100$, $1 \leq y \leq 100$, $1 \leq n \leq 100$ if this applicant is a candidate, there remain n more applicants to be interviewed, we have reached a new candidate with quality x and quality of the secretary we are possessing is y.

Let $T(x, y, n)$ be defined as the time of the first candidate after the time when we have n applicants to be seen. Let $p(x, y, n, k) = P\{T(x, y, n)=k\}$. Then it is easy to see that

$$p(x,y,n,k) = \begin{cases} \dfrac{100-x}{100}\cdot\left(\dfrac{x}{100}\right)^{k-1}, & k=1,\ 2,\ ,n \\ \left(\dfrac{x}{100}\right)^{k-1}, & k=n+1 \end{cases} \quad . \tag{3.38}$$

Let $W(x, y, n)$ be the expected additional payoff under an optimal strategy starting from state (x, y, n), $1 \le x \le 100$, $1 \le y \le 100$, $1 \le n \le 100$, and also let $U(x, y, n)$ be the expected additional payoff when we select the new candidate and then continues search in an optimal manner. Similarly, let $V(x, y, n)$ be the expected additional payoff when we reject the new candidate and then continues search in an optimal manner. Then the principle of optimality yields, for $1 \le x \le 100$, $1 \le y \le 100$, $1 \le n \le 100$

$$W(x,\ y,\ n) \ = \ max\{U(x,\ y,\ n),\ V(x,\ y,\ n)\} \tag{3.39}$$

where

$$U(x,y,n) = \sum_{k=1}^{n+1}\left(Expected\ Payoff\ if\ T(x,\ y,\ n)=k\right)\cdot p(x,y,n,k) \tag{3.40}$$

$$= \sum_{k=1}^{n}\left[\left(kx-d+\frac{1}{100-x}\int_x^{100} W(t,x,n-k)dt\right)\cdot p(x,y,n,k)\right]$$

$$+(nx-d)\cdot p(x,y,n,n+1)$$

$$= \sum_{k-1}^{n}\left[\left(kx-d+\frac{1}{100-x}\int_x^{100} W(t,x,n-k)dt\right)\cdot\frac{100-x}{100}\cdot\left(\frac{x}{100}\right)^{k-1}\right]$$

$$+(nx-d)\cdot\left(\frac{x}{100}\right)^n$$

and

$$V(x,y,n)=\sum_{k=1}^{n+1}\left(Expected\ Payoff\ if\ T(x,\ y,\ n)=k\right)\cdot p(x,y,n,k) \tag{3.41}$$

$$=\sum_{k=1}^{n}\left[\left(ky+\frac{1}{100-x}\int_x^{100}W(t,y,n-k)dt\right)\cdot p(x,y,n,k)\right]$$

$$+ny\cdot p(x,y,n,n+1)$$

$$=\sum_{k=1}^{n}\left[\left(ky+\frac{1}{100-x}\int_x^{100}W(t,y,n-k)dt\right)\cdot\frac{100-x}{100}\cdot\left(\frac{x}{100}\right)^{k-1}\right]$$

$$+ny\cdot\left(\frac{x}{100}\right)^n$$

Using the boundary condition: $U(x,y,0)=x-d$, $U(100,y,n)=100(n+1)-d$, $V(x,y,0)=y$ and $V(100,y,n)=y(n+1)$, $1\leq x\leq 100$, $1\leq y\leq 100$, $1\leq n\leq 100$, we can solve Equations (3.39)-(3.41) recursively to yield the optimal strategy and optimal value *W(x, y, n)*.

As each state of the process is described by three different variables, namely $1\leq x\leq 100$, $1\leq y\leq 100$, $1\leq n\leq 100$, there would be 10^6 different states involved in the process. In order to determine the optimal strategy associated with these 10^6 possibilities, we could employ numerical methods to obtain the solution. The following is the corresponding pseudo-code to calculate the expected additional payoff under an optimal strategy starting from state *(x, y, n)* for, $1\leq x\leq 100$, $1\leq y\leq 100$, $1\leq n\leq 100$. The replacement cost is *d=500*.

---

main function

read in the values of x, y and n

```
        calculate the values of U(x, y, n) and V(x, y, n)
        if V(x, y, n)> U(x, y, n)
            reject the applicant
        else
            select the applicant
        end if
end function

W(x, y, n)
      calculate the values of U(x, y, n) and V(x, y, n)
        if V(x, y, n)> U(x, y, n)
            return V(x, y, n)
        else
            return U(x, y, n)
        end if
end function

U(x, y, n)
      If x=100
            return 100(n+1)-d
      else if n=0
            return x-d
      else
```

$$\text{return} \quad \sum_{k=1}^{n+1} \left(\text{Expected Payoff if } T(x,\ y,\ n) = k \right) \cdot p(x, y, n, k)$$

```
      end if
end function

V(x, y, n)
      If x=100
```

return y(n+1)

else if n=0

return y

else

$$return \ \sum_{k=1}^{n+1} \left(Expected\ Payoff\ if\ T(x,\ y,\ n) = k \right) \cdot p(x, y, n, k)$$

end if

end function

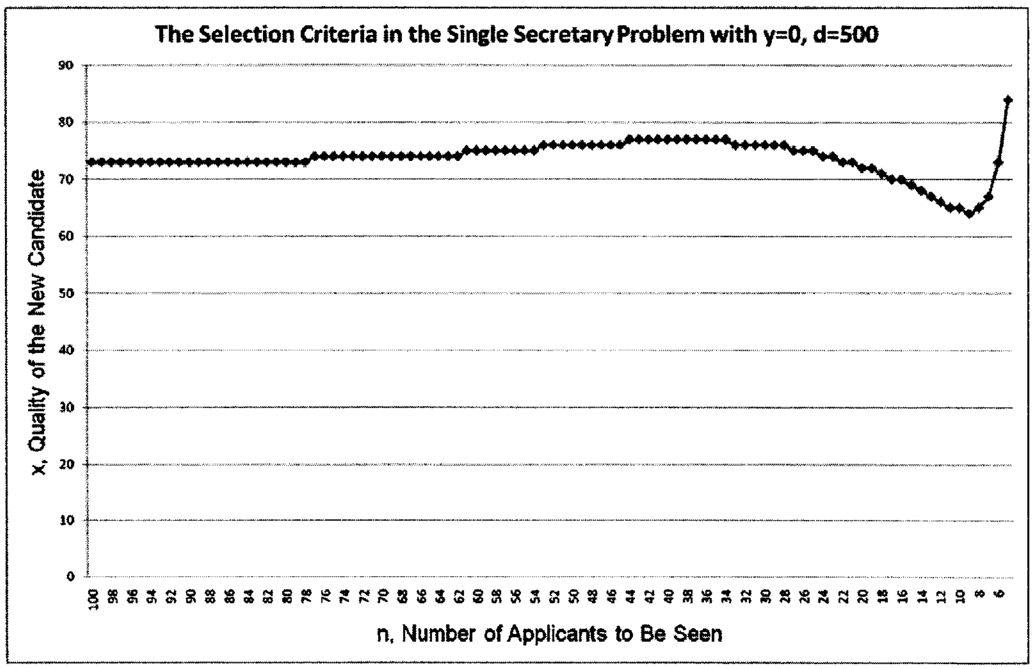

Figure 8. The Selection Criteria in the Single Secretary Problem with y=0, d=500

As there are 10^6 different possible states in the process, it is impossible to display the optimal strategies for all the possible states here. Figure 8 shows the criteria for selecting the new secretary with different numbers of remaining applicants, for the case the quality of the current secretary is $y=0$, and the replacement cost is $d=500$. We can see from the plot, that at the very beginning of the interview process, we should not select any applicant whose quality is lower than 73. The criteria increase steadily for the first 2/3 of the population, but decreases for n greater than 34. It is interesting to note that the selection criteria starts to increase dramatically when we have only 10 applicants to be observed and no selection should be made when the number of remaining applicants is 5 or less. This makes sense because when the interview process approaches its end, the benefit of changing secretaries is unlikely to outweigh the huge switch costs we have to invest.

Chapter 4. Multiple Secretary Problem with Switch Costs

In this chapter we consider another variant of the secretary problem whereby the interviewer is allowed to hire more than one secretary and that there are switch costs.. In this problem an infinite number of applicants, whose qualities are independent, identically distributed from a uniform distribution, are interviewed in sequence. Assuming that the interviewer can employ at most m secretaries at any one time, and that he receives a payoff as long as he employs at least one secretary, then the total payoff can be given as the sum of all the selected secretaries' qualities multiplied by the length of time we are in possession of these secretaries, less the total switching costs:

$$Payoff = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} quality\ of\ secretary\ j\ at\ time\ i \right) - (\ switch\ cost \times No.\ of\ switches).$$

The objective is to find the optimal policy to maximize the total payoff.

As the interviewer now has more vacancies to keep more secretaries, the selection criterion is expected to be lowered. The interviewer might make more selections along the interview process as compared to the case in the single secretary problem. Because of the complex calculation involved in this finite horizon problem, we may use dynamic programming technique and numerical methods to calculate and estimate the optimal policy.

4.1 Multiple Choice Duration Problem

Tamaki et al. (1991) attempts to extend the one choice problems to the multiple choice problems. They explored a variant of the multiple choice secretary problem that is known as the multiple choice duration problem, in which the objective is to maximize the time of possession of relatively best objects. For the m choice duration problem with a known number of objects, there exists a sequence of critical numbers

$(s_1, s_2, \ldots, s_m)$ such that, whenever there remain k choices to be made, then the optimal strategy immediately selects a relatively best object if it appears after or on time s_k, and receive each time a unit payoff as long as either of the chosen objects remains a candidate.

Obviously only candidates can be chosen, the objective being to maximize expected payoff. This problem can be viewed from another perspective as follows, Let *T(i)* be defined as the time of the first candidate after time *i* if there is one, and *n+1* if there is none. Then *T(i)-i* is the duration of the candidate selected at time *i* and the objective is to find a stopping vector $(\tau_1^*, \tau_2^*, \ldots, \tau_m^*)$ such that

$$E\left[\sum_{i=1}^{m}\left(T(\tau_i^*) - \tau_i^*\right)\right] = \sup_{(\tau_1, \tau_2, \ldots \tau_m) \in C_m} E\left[\sum_{i=1}^{m}\left(T(\tau_i) - \tau_i\right)\right] \tag{4.1}$$

where τ_i, $1 \le i \le m$, denotes the stopping time related to the ith choice and C_m is the set of all possible vectors $(\tau_1, \tau_2, \ldots, \tau_m)$.

The m choice duration problem can be considered as a Markovian decision model. Since the decision to select or reject takes place only when a candidate appears, we describe the state of the process as *(i, k)*, $1 \le i \le n$, $1 \le k \le m$ where the ith applicant is a candidate and k is the number of remaining choices to be made. For the above process to be a Markov chain, we must further introduce additional absorbing state *(n+1, k)* that denotes the situation where the process comes to an end with k choices left for $1 \le k \le m$.

The expected duration of the candidate selected in *(i, k)* is given by $E[T(i) - i]$, which is calculated as

$$E[T(i) - i] = \sum_{j=i+1}^{n+1} (j-i)\, p(i,j) = \sum_{j=i}^{n} \frac{i}{j} \tag{4.2}$$

To make the solutions of the models more easily comparable to each other, the expected contribution of the candidate selected in *(i, k)* is evaluated as $E[T(i) - i] / n$ instead of $E[T(i) - i]$.

Let $W(i,k)$ be the expected additional payoff under an optimal strategy starting from state *(i, k)*, $1 \le i \le n$, $1 \le k \le m$, and also let $U(i,k)$ and $V(i,k)$ be the expected additional payoff when we select and reject the *i*th object and then continues search in an optimal manner. Then the principle of optimality yields, for $1 \le i \le n$ and $1 \le k \le m$

$$W(i,k) = max\{U(i,k), V(i,k)\}, \tag{4.3}$$

where

$$U(i,k) = E\left[\frac{T(i)-i}{n} + W(T(i),k-1)\right] = \frac{1}{n}\sum_{j=i}^{n}\frac{i}{j} + \sum_{j=i+1}^{n}\frac{i}{j(j-1)}\cdot W(j,k-1) \tag{4.4}$$

and

$$V(i,k) = W(T(i),k) = \sum_{j=i+1}^{n}\frac{i}{j(j-1)}W(i,k). \tag{4.5}$$

There equations can be solved recursively to yield the optimal strategy and the optimal value $W(1,m)$. The optimal strategy can be summarized as follows. There exists a sequence of integer-valued critical numbers $(s_1, s_2, \ldots, s_m)$ such that the optimal strategy immediately selects a candidate if it appears at or after time s_k:

$$s_k = min\{i : G(i,k) \ge 0\} \tag{4.6}$$

where

$$G(i,k) = G(i,1) + \sum_{j=\max(i+1,s_{k-1})}^{n}\frac{1}{j-1}\cdot G(j,k-1) \tag{4.7}$$

and

$$G(i,1) = \sum_{j=i}^{n}\frac{1}{j} - \sum_{j=i+1}^{n}\frac{1}{j-1}\sum_{t=j}^{n}\frac{1}{t}. \tag{4.8}$$

4.2 Dynamic Programming

Dynamic programming, like the divide-and-conquer method, solves problems by combining the solutions to subproblems. Dynamic programming is still applicable even when the subproblems are not independent. The divide-and-conquer algorithm is inefficient as it treats each subproblem uniquely and would often solve the same subproblem multiple times. Dynamic programming on the other hand, stores the result of each unique subproblem on a lookup table and simply retrieves this result from the lookup table if a subproblem that has already been solved is encountered.

Dynamic programming is typically applied to optimization problems. In such problems there can be many possible solutions. In the multiple secretary problem with switch cost, we wish to find an optimal policy to achieve the optimal value, that is, the maximum payoff.

There is a variation of dynamic programming that offers the efficiency of the usual dynamic programming approach while maintaining a top-down strategy. The idea is to memorize the natural, but inefficient, recursive algorithm. As in ordinary dynamic programming, we maintain a table with subproblem solutions, but the control structure for filling in the table is more like the recursive algorithm. A memorized recursive algorithm maintains an entry in a table for the solution to each subproblem. Each table entry initially contains a special value to indicate that the entry has yet to be filled in. When the subproblem is first encountered during the execution of the recursive algorithm, its solution is computed and then stored in the table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned.

To solve a complicated problem like multiple secretary problem with switch cost, the natural recursive algorithm without memorization runs in exponential time since solved subproblem are repeatedly solved. Memoization provides a more efficient way to determine the optimal rules.

4.3 Numerical Analysis

In the multiple secretary problem with switch costs, the interviewer can have at most m secretaries at a time, thus there will be more variables involved in describing the Markov decision process model. We describe the state of the process as $(x, y_1, \dots y_m, n)$, $1 \le x \le 100$, $1 \le y \le 100$, $1 \le n \le 100$ where x is the quality of the candidate, n is the remaining number of applicants to be observed, and $y_1, y_2, \dots, y_m$ are the qualities of the current secretaries. We have $y_1 \le y_2 \le \dots \le y_m$.

Let $W(x, y_1, \dots, y_m, n)$ be the expected additional payoff under an optimal strategy starting from state $(x, y_1, \dots, y_m, n)$, $1 \le x \le 100$, $1 \le y \le 100$, $1 \le n \le 100$, $y_1 \le y_2 \le \cdots \le y_m$, and also let $U(x, y_1, \dots, y_m, n)$ be the expected additional payoff when we select the new candidate and then continue search in an optimal manner. Similarly, let $V(x, y_1, \dots, y_m, n)$ be the expected additional payoff when we reject the new candidate and then continue search in an optimal manner. Then the principle of optimality yields, for $1 \le x \le 100$, $1 \le y \le 100$, $1 \le n \le 100$

$$W(x, y_1, \dots, y_m, n) = max\{U(x, y_1, \dots, y_m, n), \ V(x, y_1, \dots, y_m, n)\} \tag{4.9}$$

where

$$U(x, y_1, \dots, y_m, n) \tag{5.0}$$

$$= \sum_{k=1}^{n+1} \left(Expected\ Payoff\ if\ T(x, y_1, \dots, y_m, n) = k \right) \cdot p(x, y_1, \dots, y_m, n, k)$$

$$= \sum_{k=1}^{n} \left(k(x + y_2 + \cdots + y_m) - d + \frac{1}{100 - x} \int_x^{100} W(t, x, y_2, \dots, y_m, n-k)\, dt \right)$$

$$\times p(x, y_1, \ldots, y_m, n, k) + [n(x + y_2 + \cdots + y_m) - d] \cdot p(x, y_1, \ldots, y_m, n, n+1)$$

$$= \sum_{k=1}^{n} \left(k(x + y_2 + \cdots + y_m) - d + \frac{1}{100 - x} \int_x^{100} W(t, x, y_2, \ldots, y_m, n-k) dt \right)$$

$$\times \frac{100 - min(x, y_2)}{100} \cdot \left(\frac{min(x, y_2)}{100} \right)^{k-1}$$

$$+ [n(x + y_2 + \cdots + y_m) - d] \cdot \left(\frac{min(x, y_2)}{100} \right)^n$$

and

$$V(x, y_1, \ldots, y_m, n) \tag{5.1}$$

$$= \sum_{k=1}^{n+1} (Expected\ Payoff\ if\ T(x,\ y_1, \ldots, y_m,\ n) = k) \cdot p(x, y_1, \ldots, y_m, n, k)$$

$$= \sum_{k=1}^{n} \left(k(y_1 + \cdots + y_m) + \frac{1}{100 - x} \int_x^{100} W(t, y_1, \ldots, y_m, n-k) dt \right)$$

$$\times p(x, y_1, \ldots, y_m, n, k) + n(y_1 + \cdots + y_m) \cdot p(x, y_1, \ldots, y_m, n, n+1)$$

$$= \sum_{k=1}^{n} \left(k(y_1 + \cdots + y_m) + \frac{1}{100 - x} \int_x^{100} W(t, y_1, \ldots, y_m, n-k) dt \right)$$

$$\times \frac{100 - x}{100} \cdot \left(\frac{x}{100} \right)^{k-1} + n(y_1 + \cdots + y_m) \cdot \left(\frac{x}{100} \right)^n$$

Equations (4.9)-(5.1), combined with the boundary conditions $U(x, y_1, \ldots, y_m, 0) = x + y_1 + \ldots + y_m - d$, $U(100, y_1, 100, \ldots, y_m, n) = m \times 100(n+1) - d$, $V(100, y_1, \ldots, y_m, 0) = (y_1 + \ldots + y_m) \times (n+1)$ and $V(x, y_1, \ldots, y_m, 0) = y_1 + \ldots + y_m$, $1 \le x \le 100$, $1 \le y \le 100$, $1 \le n \le 100$, $y_1 \le y_2 \le \cdots \le y_m$, can be solved recursively to yield the optimal strategy and optimal value $W(x,\ y_1, \ldots, y_m,\ n)$.

As each state of the process is described by $m+2$ different variables, namely

$1 \le x \le 100$, $1 \le y_1 \le y_2 \le \cdots \le y_m \le 100$, $1 \le n \le 100$, there could be 10^{2m+4} different states involved in the process. In order to determine the optimal strategy associated with these 10^{2m+4} possibilities, we could employ numerical methods and dynamic programming algorithm to obtain the solution. To solve the problem more efficiently, memorization method is exploited and a matrix is constructed to store subproblem solutions. The following is the corresponding pseudo-code to calculate the expected additional payoff under an optimal strategy starting from state $(x, y_1, \ldots, y_m, n)$, $1 \le x \le 100$, $1 \le y \le 100$, $1 \le n \le 100$ and $y_1 \le y_2 \le \cdots \le y_m$. The switch cost in the simulation is set to *d=500*.

---

construct a global matrix $M[x, y_1, \ldots, y_m, n]$

main function

read in the values of $x, y_1, \ldots, y_m, n$

assign a special value to the matrix $M[x, y_1, \ldots, y_m, n] \leftarrow \infty$

calculate the values of $U(x, y_1, \ldots, y_m, n)$ *and* $V(x, y_1, \ldots, y_m, n)$

if $V(x, y_1, \ldots, y_m, n) > U(x, y_1, \ldots, y_m, n)$

$M[x, y_1, \ldots, y_m, n] \leftarrow V(x, y_1, \ldots, y_m, n)$

reject the applicant

else

$M[x, y_1, \ldots, y_m, n] \leftarrow U(x, y_1, \ldots, y_m, n)$

select the applicant

end if

end function

$W(x, y_1, \ldots, y_m, n)$

if $M[x, y_1, \ldots, y_m, n] < \infty$

return $M[x, y_1, \ldots, y_m, n]$

else

calculate the values of $U(x, y_1, \ldots, y_m, n)$ *and* $V(x, y_1, \ldots, y_m, n)$

if $V(x, y_1, \ldots, y_m, n) > U(x, y_1, \ldots, y_m, n)$

$M[x, y_1, \ldots, y_m, n] \leftarrow V(x, y_1, \ldots, y_m, n)$

return $V(x, y_1, \ldots, y_m, n)$

else

$M[x, y_1, \ldots, y_m, n] \leftarrow U(x, y_1, \ldots, y_m, n)$

return $U(x, y_1, \ldots, y_m, n)$

end if

end if

end function

$U(x, y_1, \ldots, y_m, n)$

If $x = y_2 = 100$

return $m \times 100(n+1) - d$

else if $n=0$

return $x + y_1 + \ldots + y_m - d$

else

return

$$\sum_{k=1}^{n} \left(k(x + y_2 + \cdots + y_m) - d + \frac{1}{100 - x} \int_x^{100} W(t, x, y_2, \ldots, y_m, n-k)\, dt \right)$$

$$\times \frac{100-min(x,y_2)}{100} \cdot \left(\frac{min(x,y_2)}{100}\right)^{k-1}$$

$$+\left[n(x+y_2+\cdots+y_m)-d\right] \cdot \left(\frac{min(x,y_2)}{100}\right)^{n}$$

end if

end function

V(x, y, n)

If x=100

return $(y_1+\ldots+y_m)\times(n+1)$

else if n=0

return $y_1+\ldots+y_m$

else

return

$$\sum_{k=1}^{n}\left(k(y_1+\cdots+y_m)+\frac{1}{100-x}\int_x^{100} W(t,y_1,\ldots,y_m,n-k)\,dt\right)$$

$$\times \frac{100-x}{100} \cdot \left(\frac{x}{100}\right)^{k-1} + n(y_1+\cdots+y_m) \cdot \left(\frac{x}{100}\right)^{n}$$

end if

end function

4.3.1 The Two Secretary Case

The interviewer can have at most 2 secretaries at any time, hence each state of the process is described by 4 variables and there would be 10^8 different states. The optimal additional payoff at each state can be represented by a 32-bit floating point value, thus the minimum space requirement is 3.2×10^9 bits, which is 3.2 gigabits.

This still exceeds the available computer memory size.

Thus, instead of keeping the subproblem solutions in a matrix, we can create a file in the hard disc that maintains an entry per subproblem with the solution to each subproblem. As illustrated in Figure 9, a data structure is constructed to describe each Markov process state. Each data structure contains values of $y_1, \dots, y_m, x, n$ and the expected additional payoff under an optimal strategy starting from state $(x, y_1, \dots, y_m, n)$. These information are well organized in the file in such a way that the program can easily find one particular data set according to state variables $y_1, \dots, y_m, x, n$ and obtain the value of $W(x, y_1, \dots, y_m, n)$.

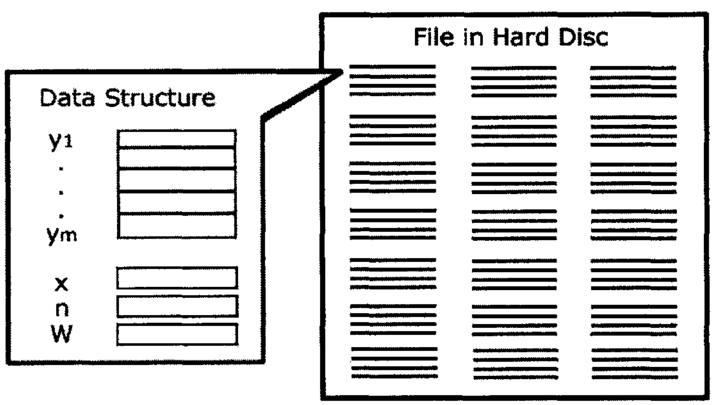

Figure 9. The Data Structure to Store the Solution to Each Subproblem

The expected additional payoff under an optimal strategy starting from each state can be calculated by using numerical simulation. Given that the interviewer has not selected any secretaries yet (i.e. $y_1 = y_2 = 0$), Figure 10 presents the threshold values of the first secretary for various values of n. It can be seen that the threshold value remains between 58 and 59 for n larger than 39; increases as n decreases from 39 to 29; and decreases as n decreases further. The threshold value increases as n decreases from 39 to 29 because the interviewer is unlikely to have the time for further replacement and thus have to select a candidate for sufficient quality. The threshold value goes down when the interview process approaches the end since the interviewer has no choice but to be satisfied with a mediocre applicant.

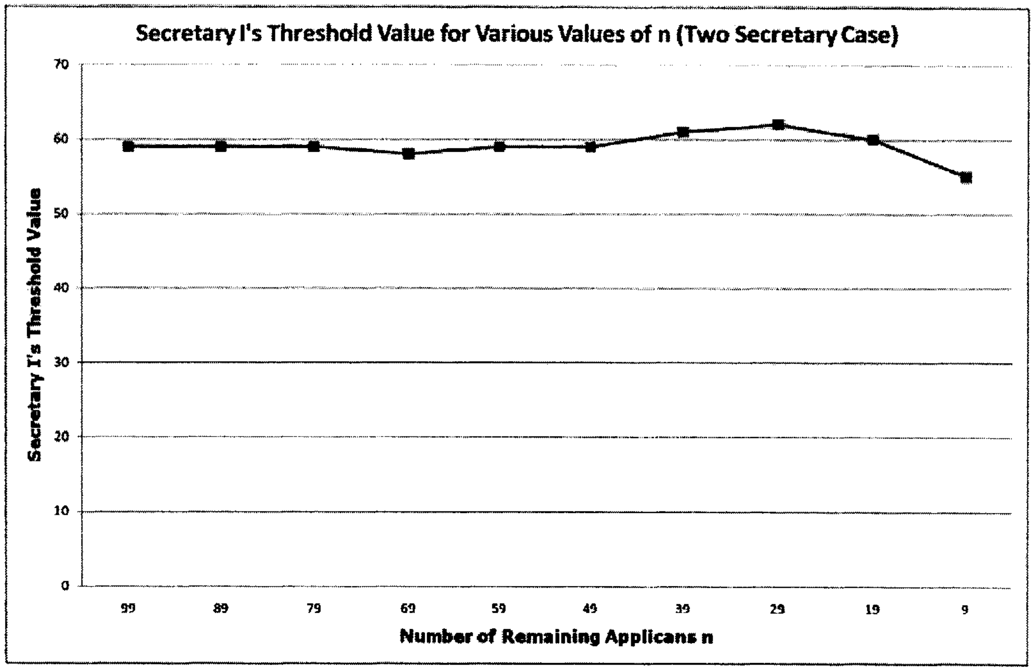

Figure 10. Secretary I's Threshold Values for Various Values of n

Figure 11 shows the selection criteria for the second secretary when we have already hired the first secretary and there remain 99 more applicants to be observed. The threshold value for the second secretary's quality increases as the quality of the first secretary increases from 0 to 78. This is because when the quality of the first secretary is higher, the interviewer is more patient and he or she would expect to get a better second secretary. The curve begins to go down after the quality of the first secretary reaches 78, and converges to a steady state value of 73. The existence of the hump can by explained by the fact that when we are holding a secretary with a sufficiently high quality, a lower quality second secretary can be tolerated because we may still have a chance to replace this secretary later on to achieve optimal payoff.

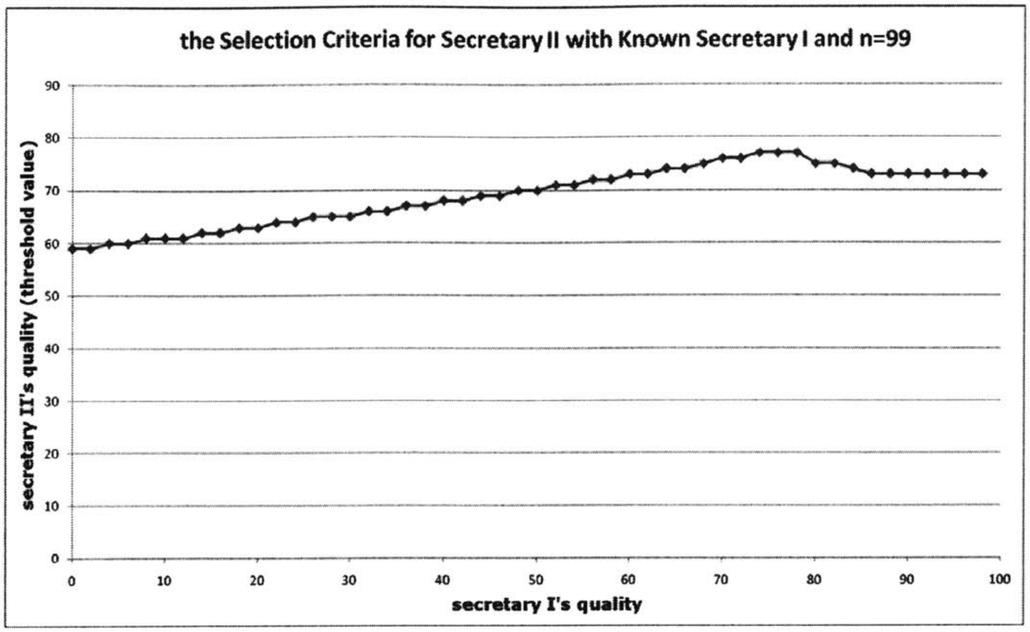

Figure 11. the Selection Criteria for Secretary II with Known Secretary I and n=99

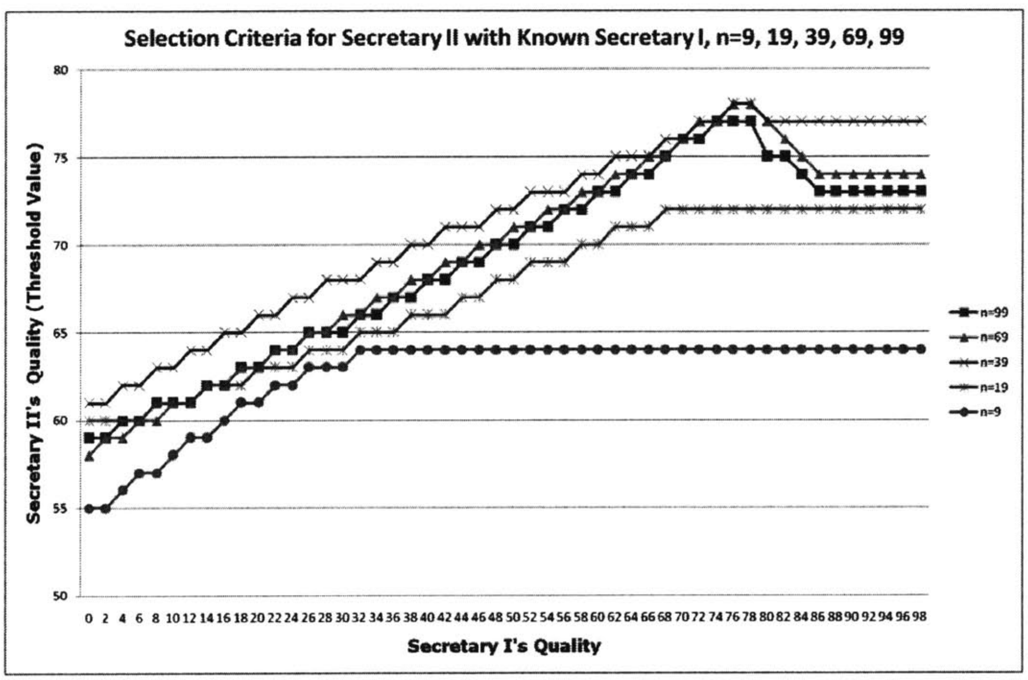

Figure 12. the Selection Criteria for Secretary II with Known Secretary I, n=9, 19, 39, 69, 99

Figure 12 shows the selection criteria for the second secretary at different time horizons. According to this plot, the hump is more obvious when we have more remaining applicants to see, and it becomes less pronounced and finally disappears when we approach the end of the interview process (n=19, 9). This is probably due to the fact that when the number of applicants is small, the possibility to change the second secretary at a later stage is low, thus the interviewer may stick to his previous selection criteria and would not accept any compromise.

4.3.2 The Seven Secretary Case

In this case the interviewer can have at most 7 secretaries at a time. If the quality of each applicant is uniformly distributed over $[0, 100]$, each state of the process should be described by 9 variables and there are 10^{18} different states. The optimal additional payoff at each state can be represented by a 32-bit floating point value, thus the space requirement is 32 exabits, which far exceeds the sizes of the disc drives currently available.

Though it is impossible to determine the optimal strategy for the case in which the quality of each applicant is uniformly distributed over $[0, 100]$, the space requirement can be significantly reduced if we change the distribution range to $[0, 5]$. Correspondingly the switch cost is rescaled to 25. By doing so, we are able to get an approximate optimal strategy for the seven secretary case.

Given that the switch cost is $d=25$, and the interviewer has not selected any secretary yet (i.e. $y_1 = y_2 = \cdots = y_7 = 0$). Table 3 gives a selection of the first secretary's threshold values x^* for various values of n. The corresponding expected additional payoff is U when we select the new candidate and is V when we reject the new candidate. This is shown in this table. It can be seen that the time horizon value n does not have a significant effect on the first secretary's threshold value, which equals to 2 consistently. It may be attributed to the approximate methodology we employ in the analysis,. Another reason is that an increase in the number of vacancies leads to a lower selection criteria, and thus, the switch cost becomes less influential.

x^*	n	U	V
2	99	1767	1794
2	89	1557	1584
2	79	1347	1374
2	69	1137	1164
2	59	927	954
2	49	717	744
2	39	508	535
2	29	308	337
2	19	126	157
2	9	10	16

Table 3. Approximate First Secretary's Threshold Values x^* for Various Values of n ($d=25$) In the Seven Secretary Case.

4.4 Experimental Results

In this section, we compare the respondents' decisions made in the experiment with the previously-derived approximate optimal selection rules. Upon arrival at the lab, respondents were seated individually and given instructions for the experiment. All respondents received instructions emphasizing their goal in the experiment – to accumulate as much total payoff by the end of the experiment.

As a metaphor for the interview process, a computer program was created with a decision-making task. The experiment comprised 100 stages, during each of which the respondent would be shown a number. Each number that appeared during one of the 100 stages was randomly generated from a discrete uniform distribution with the range [1, 100]. Respondents also had to manage an inventory that could store at most 7 numbers. Accumulating payoff was done by selecting numbers to add to their inventory. At the end of every stage, a stage payoff equal to the sum of the numbers in their inventory would be added to their total payoff. In addition, respondents could attempt to increase their payoff at the end of any stage by selecting the number that had appeared to replace one of the current numbers in their inventory. However, replacing a number would cost the respondent a fixed sum of 500. Thus, they had to trade off the benefit of replacing a number against the fixed cost that would be incurred. If we assume that the numbers that appear at each stage represent the qualities of different applicants, then the analogy between the experimental program and the problem presented should be apparent.

In order to compare the performance of the respondents with our approximate optimal selection rule, we take a look at 13 sets of experimental results in which necessary experimental information had been properly recorded and the instructions had been apparently followed.

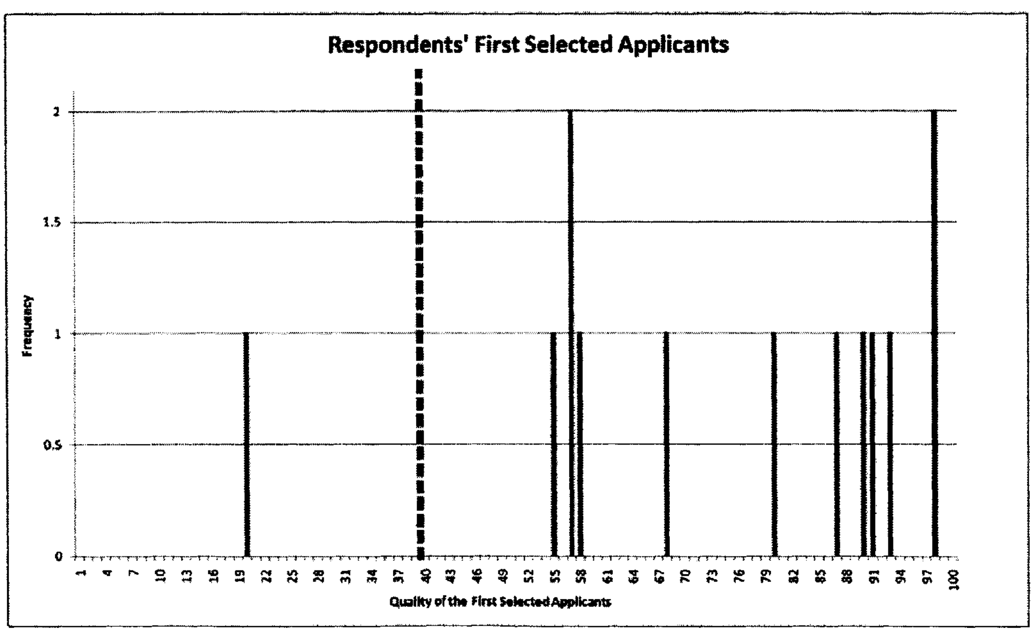

Figure 13. The Respondents' First Selected Applicants

It is worth contrasting the behavior of the respondents at the very beginning of the sequential search task to an optimal strategy benchmark, which in this experiment is to select the first number according to our simulation results in 4.3.2. Based on the approximate optimal selection rule, the optimal threshold value for the first secretary's quality is equal to 2/5 of the maximum value, which is 100 in this experiment. In other words, the first number that a respondent selects should be no less than 40. Figure 13 shows the qualities of the respondents' first selected applicants. Of the 13 respondents, 12 selected their first numbers greater than 40, and only one chose 20 as his or her initial selection. The average value of respondents' first selected numbers is 73.23, which is somewhat surprisingly high.

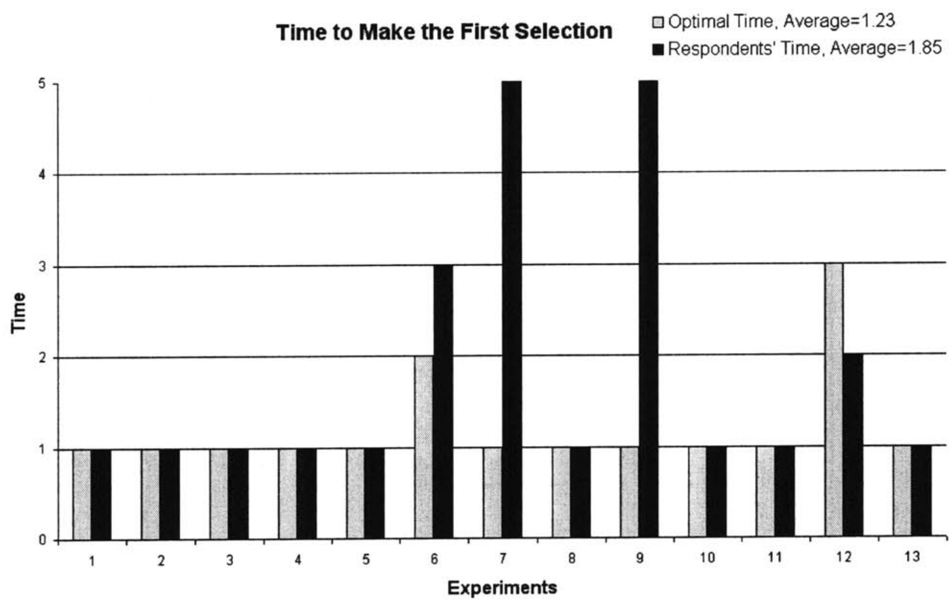

Figure 14. Time to Make the First Selection

Figure 14 compares the time to select the first secretary based on the optimal selection rule with the observed time at which the respondents made their first selections during the experiments. While most of the respondents selected their first numbers at the very beginning of the process, which is consistent with the optimal selection strategy, some of respondents waited a longer time for their first selections than the optimal benchmark does. Consequently, the average time to make the first selection under optimal rule (1.23) is shorter than that made by the respondents (1.85).

Figure 15 presents the quality variance of the selected secretaries according to the optimal selection rule and that based on the respondents' selections. The quality variance of the optimal selections is 212.3, which is smaller than the quality variance of the respondents' selections (246.7). This can be attributed to the fact that majority of the selections made by respondents take place during the first half of the process, at which the selected numbers yield a greater variance. However, the optimal rule suggests making fewer selections at early stages and as a result, the decreased number of relatively low quality secretaries reduces the total variance.

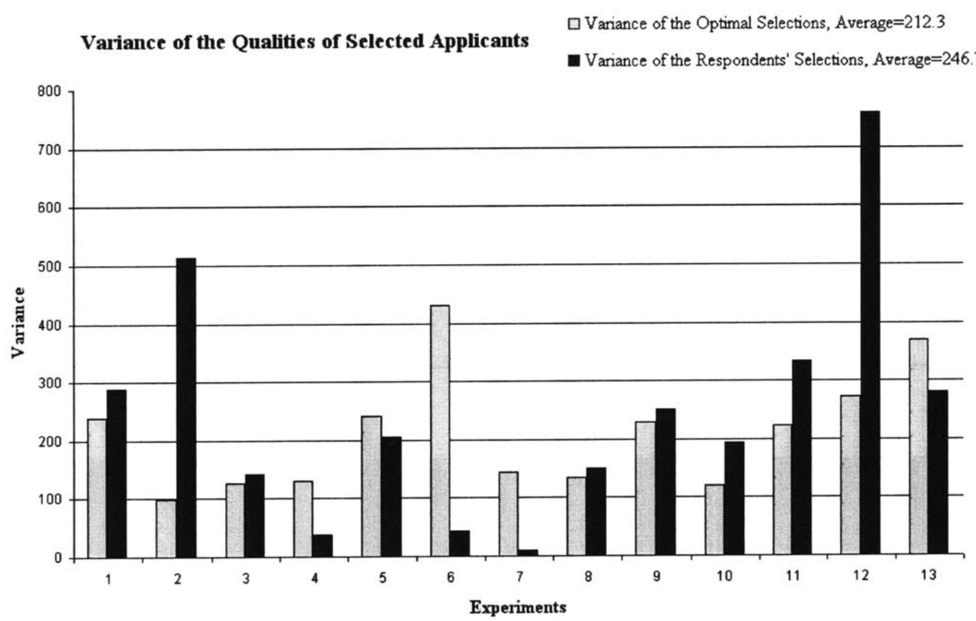

Figure 15. Variance of the Qualities of Selected Applicants

Figure 16-41 show the comparison between our selections based on the approximate optimal selection strategy and all the 13 respondents' selections. The dots in the graphs represent the values of the numbers respondents have seen along the sequential search process. Of all the 100 dots, the square shaped ones refer to the numbers selected based on the optimal strategy or by respondents respectively. Note that the motivation for switching is highly influenced by two factors – the expected number of possible replacements to be made and potential loss associated with unnecessary switches, both of which reduce the incentive to switch numbers. First, towards the end of the experiment, respondents have bigger numbers and thus a reduced need to take the risk to make the replacements. Second, the expected benefit of switching is reduced with time because the time horizon during which the new numbers can be kept is reduced. Thus it is not surprising to see that most of the selections are made during the first 1/3 of the experiment and there is a decreased tendency to select numbers later in the process. In contrast to the optimal behavior, respondents tend to make more selections or replacements at the beginning and make less later on. It is probably due to the fact that without a precise calculation, redundant early selections made by respondents lead to significant losses.

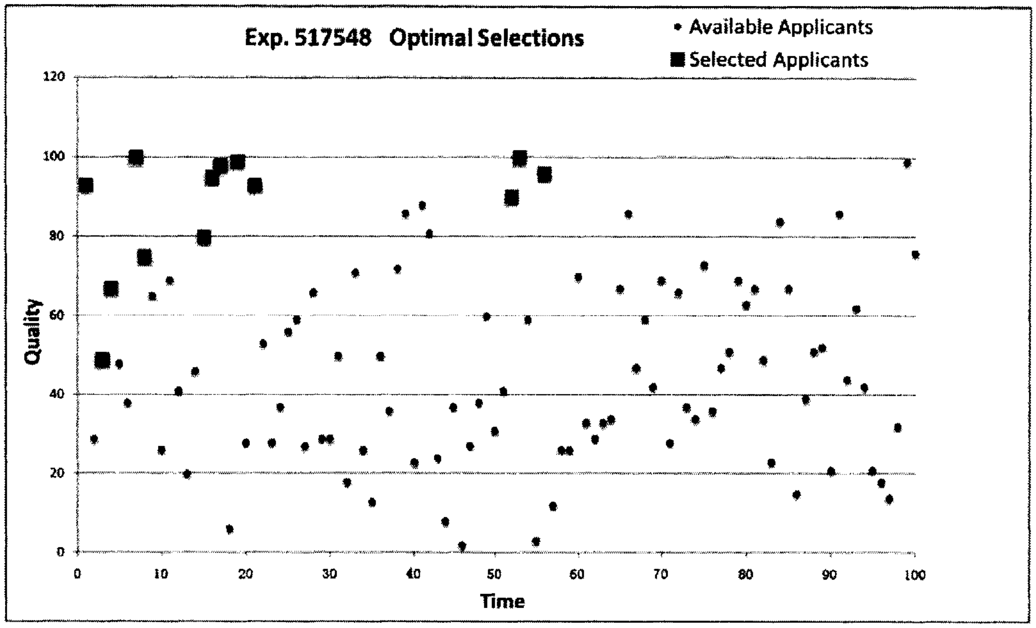

Figure 16. Selections Based on the Optimal Selection Strategy, Experiment ID: 517548

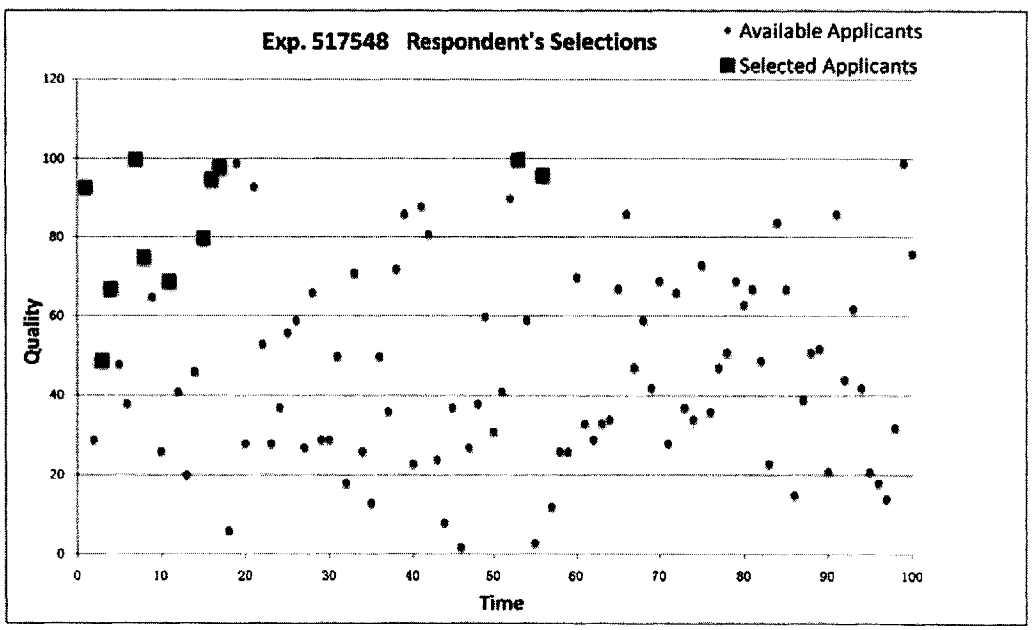

Figure 17. Respondent's Selections, Experiment ID: 517548

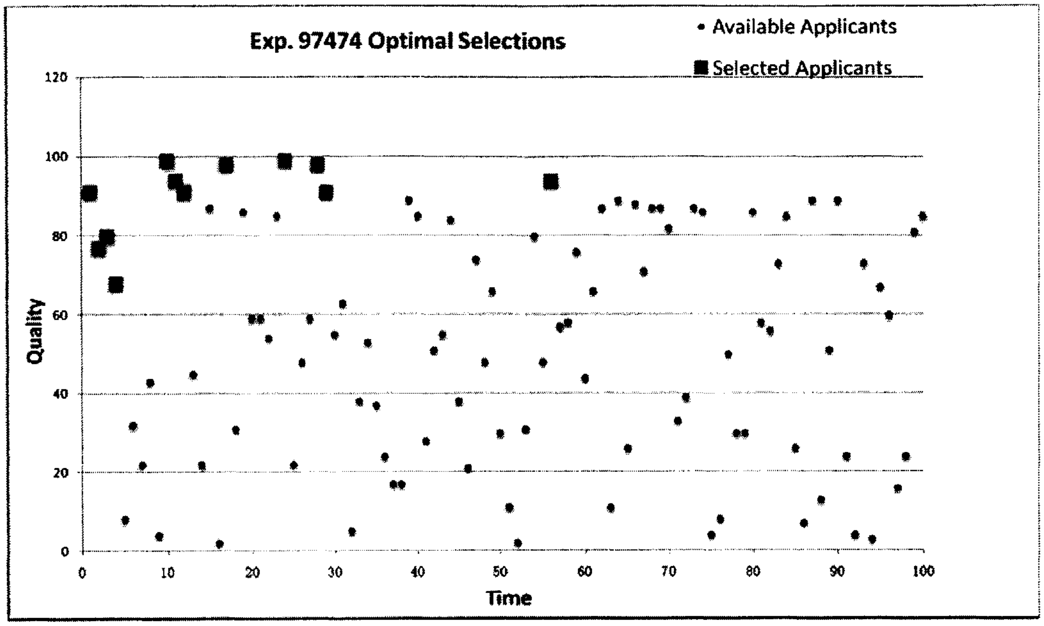

Figure 18. Selections Based on the Optimal Selection Strategy, Experiment ID: 97474

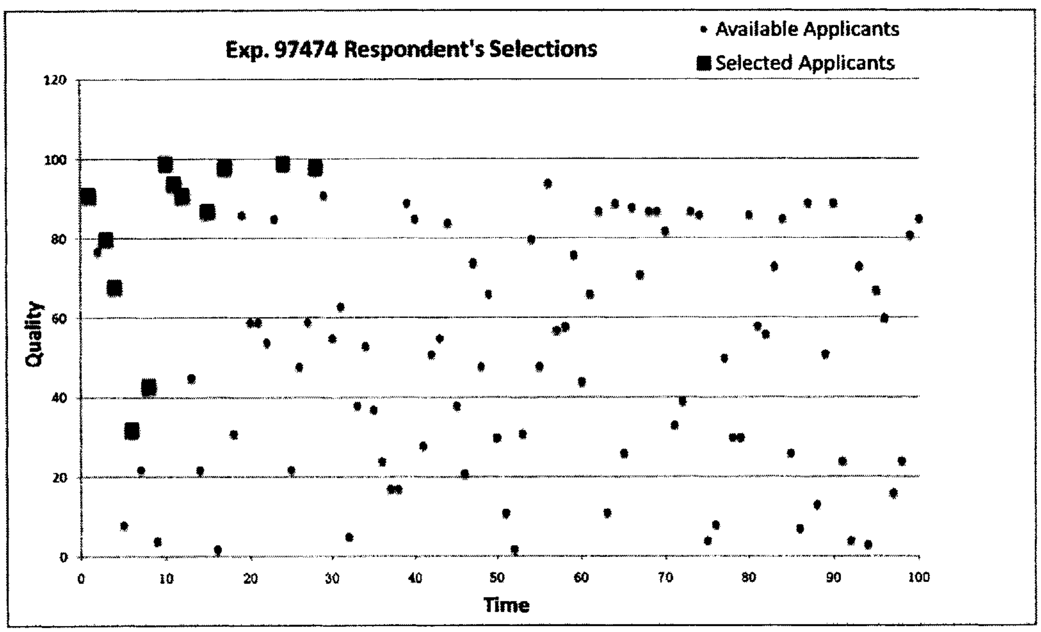

Figure 19. Respondent's Selections, Experiment ID: 97474

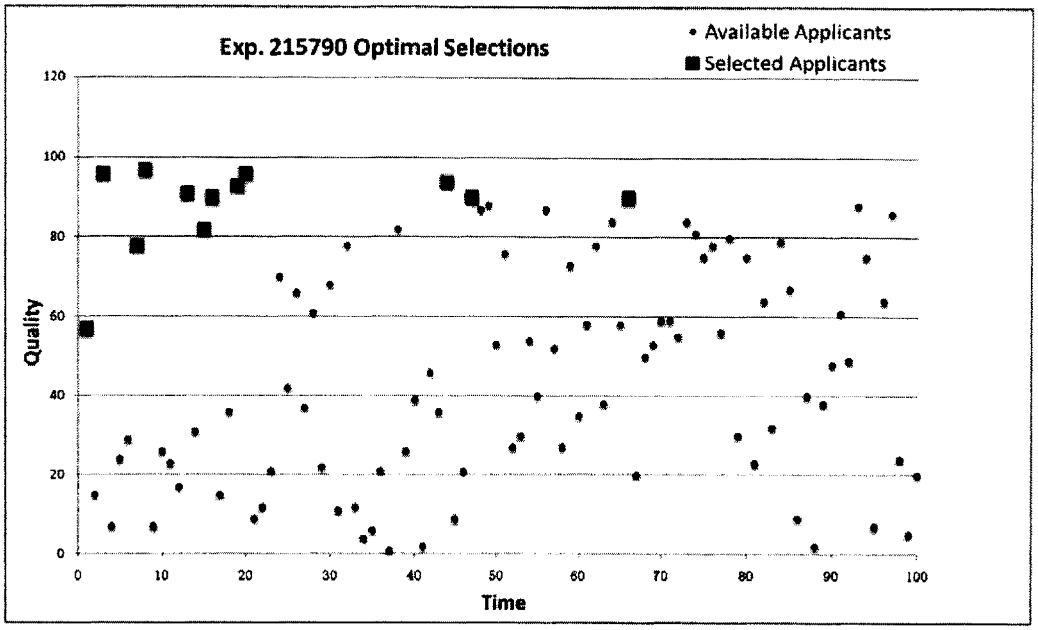

Figure 20. Selections Based on the Optimal Selection Strategy, Experiment ID: 215790

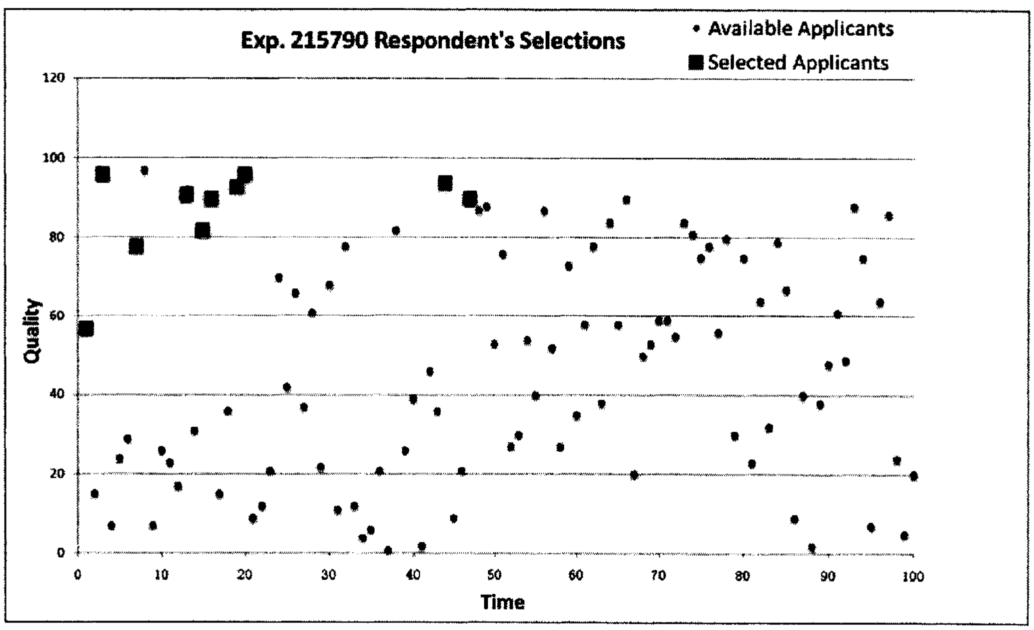

Figure 21. Respondent's Selections, Experiment ID: 215790

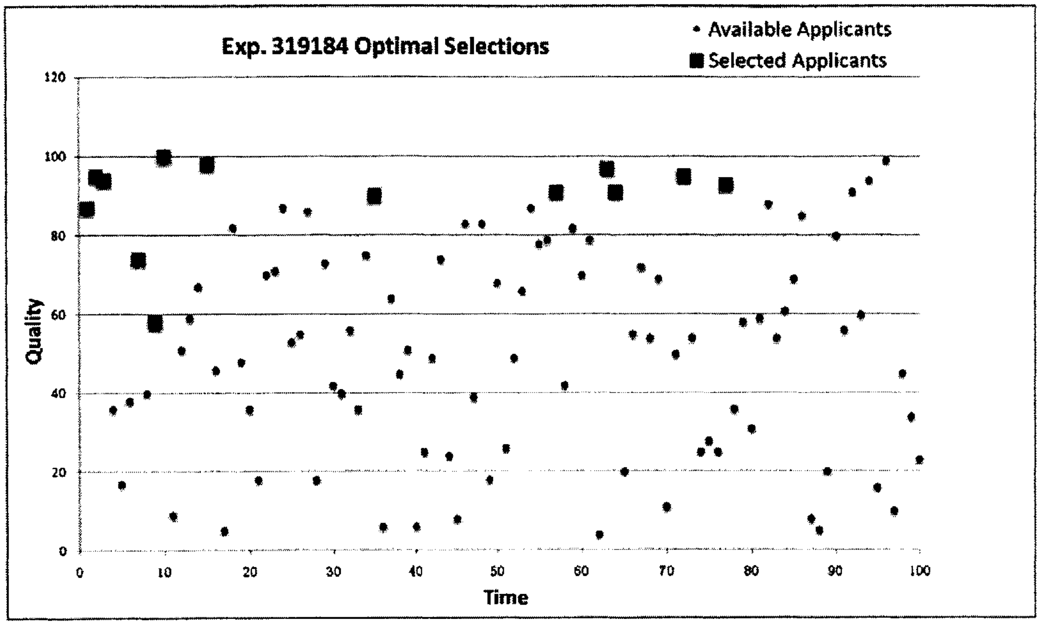

Figure 22. Selections Based on the Optimal Selection Strategy, Experiment ID: 319184

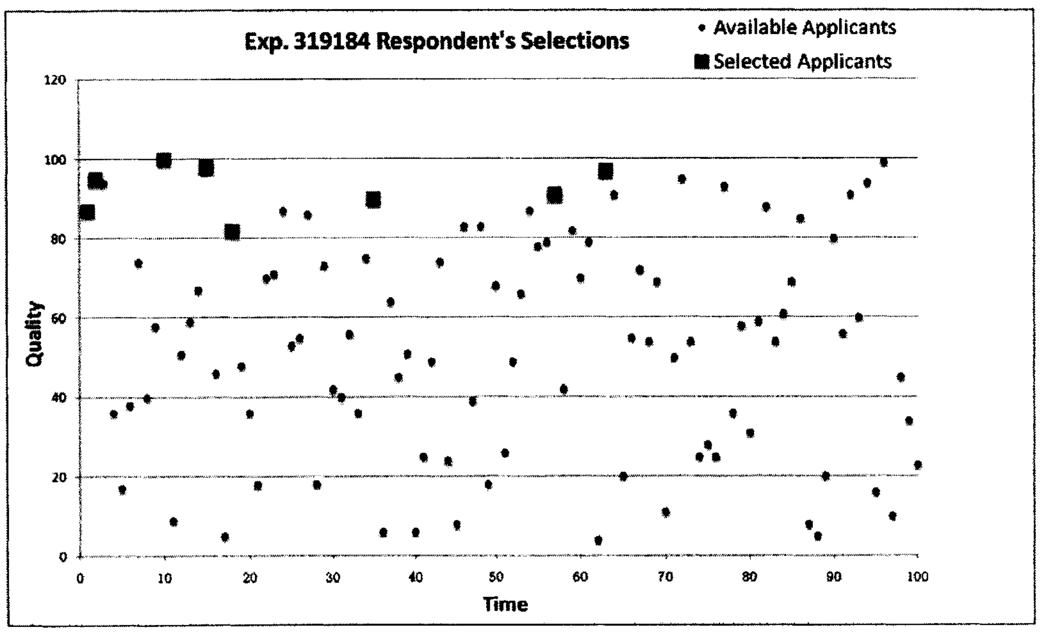

Figure 23. Respondent's Selections, Experiment ID: 319184

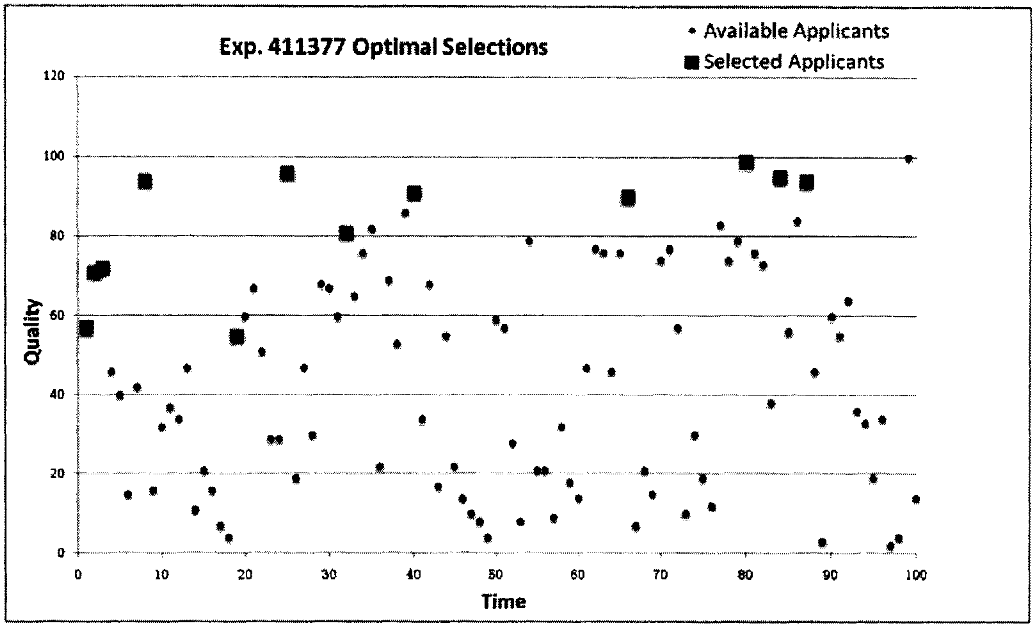

Figure 24. Selections Based on the Optimal Selection Strategy, Experiment ID: 411377

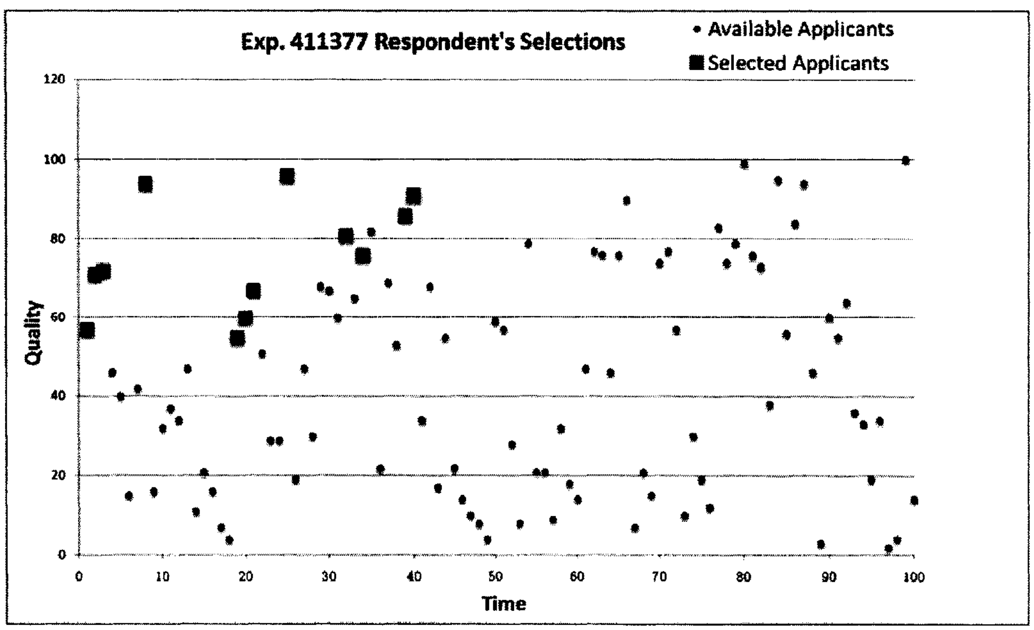

Figure 25. Respondent's Selections, Experiment ID: 411377

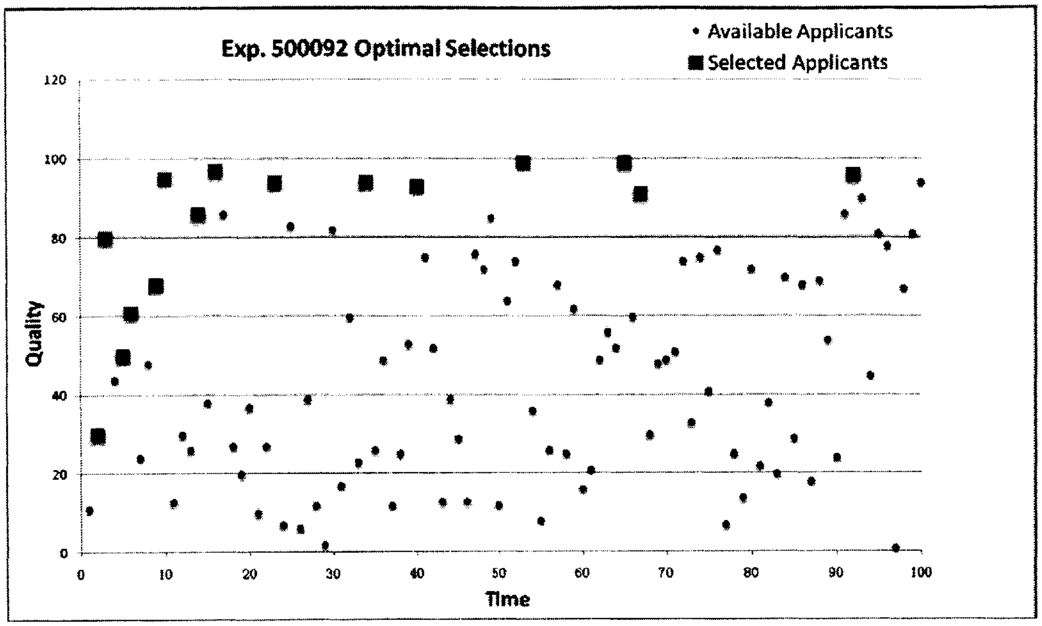

Figure 26. Selections Based on the Optimal Selection Strategy, Experiment ID: 500092

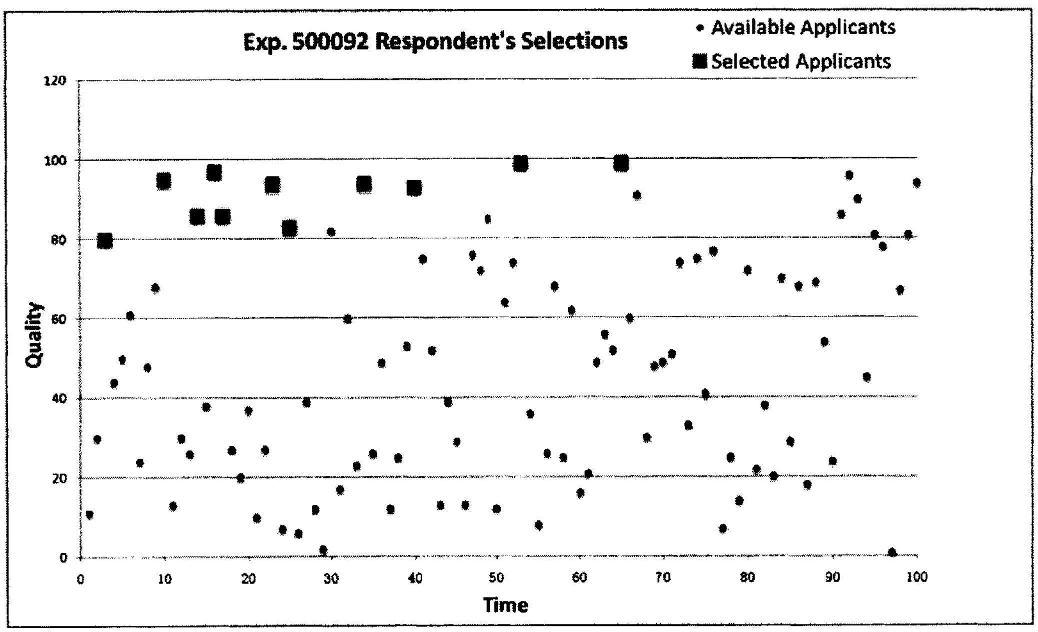

Figure 27. Respondent's Selections, Experiment ID: 500092

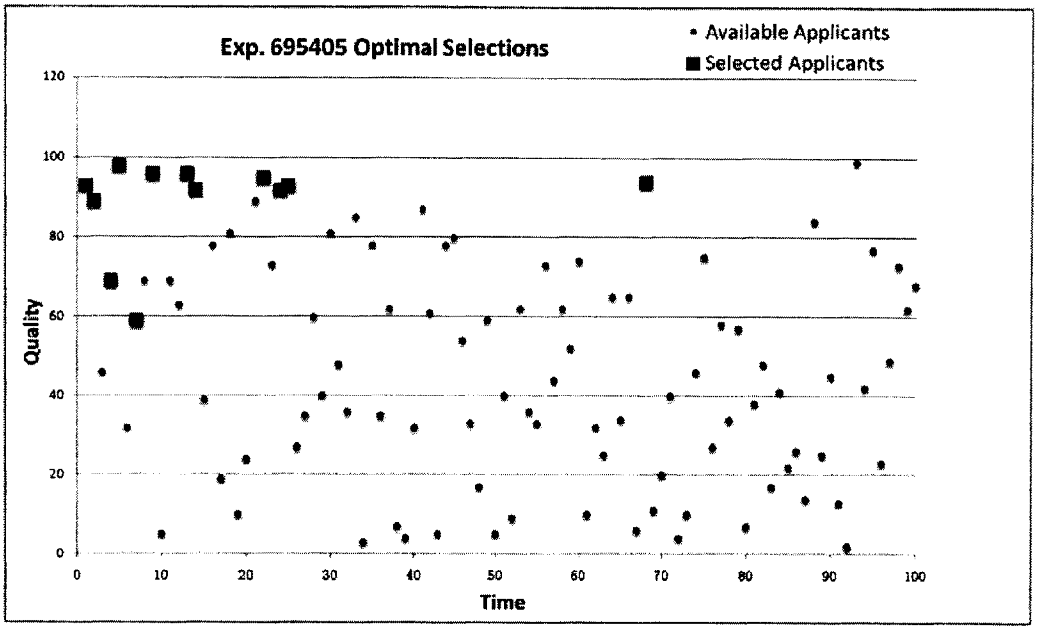

Figure 28. Selections Based on the Optimal Selection Strategy, Experiment ID: 695405

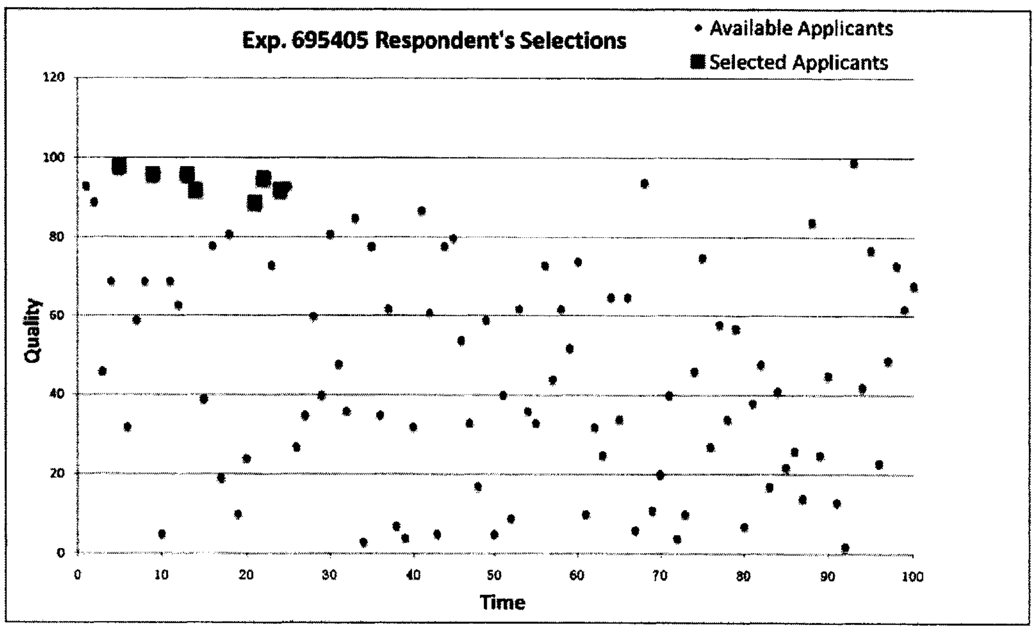

Figure 29. Respondent's Selections, Experiment ID: 695405

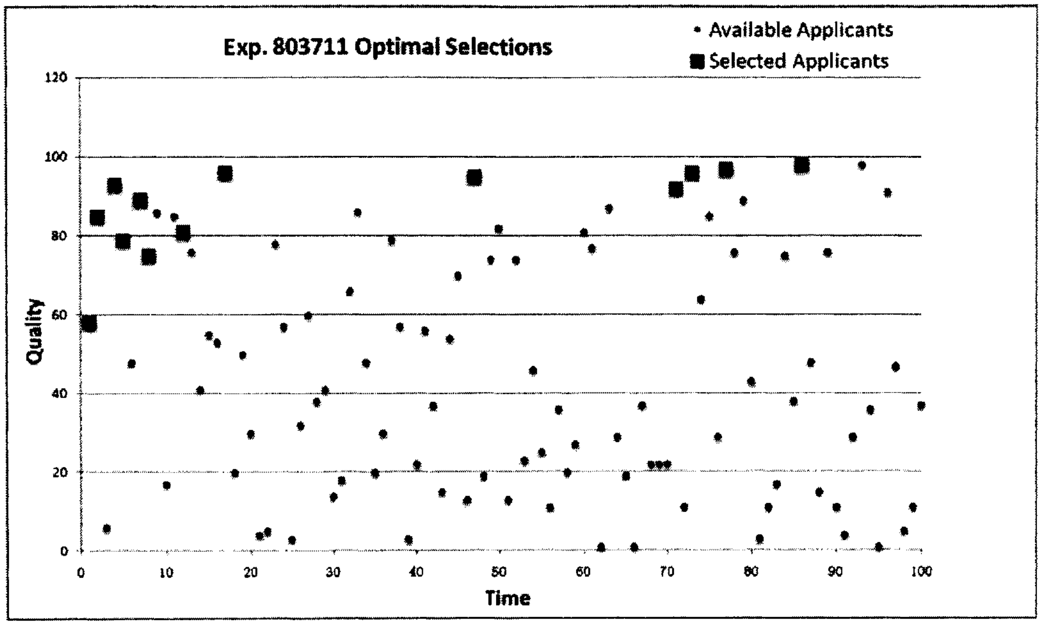

Figure 30. Selections Based on the Optimal Selection Strategy, Experiment ID: 803711

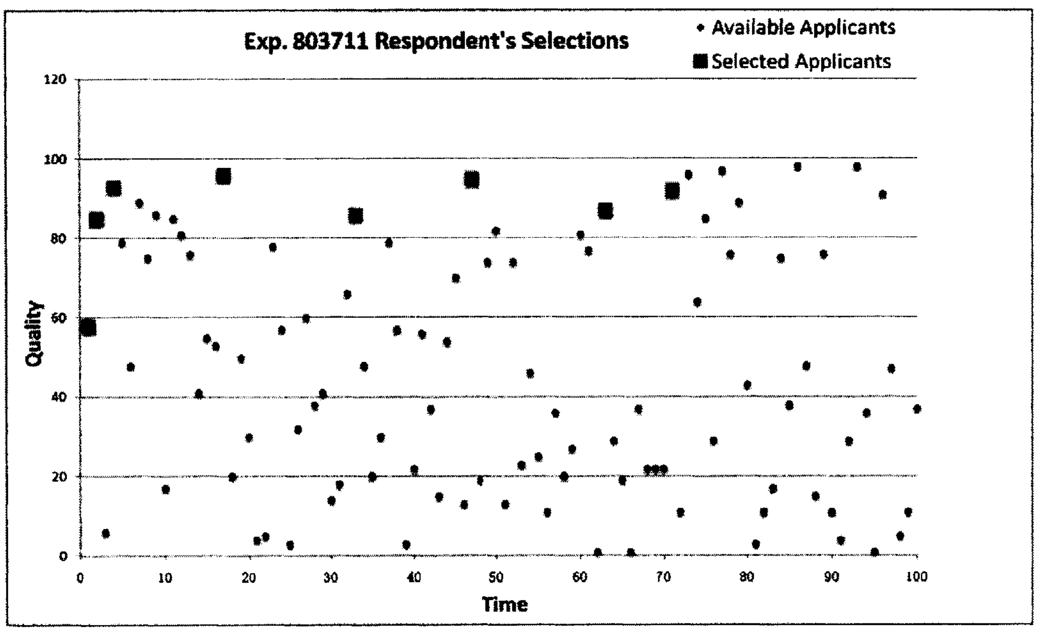

Figure 31. Respondent's Selections, Experiment ID: 803711

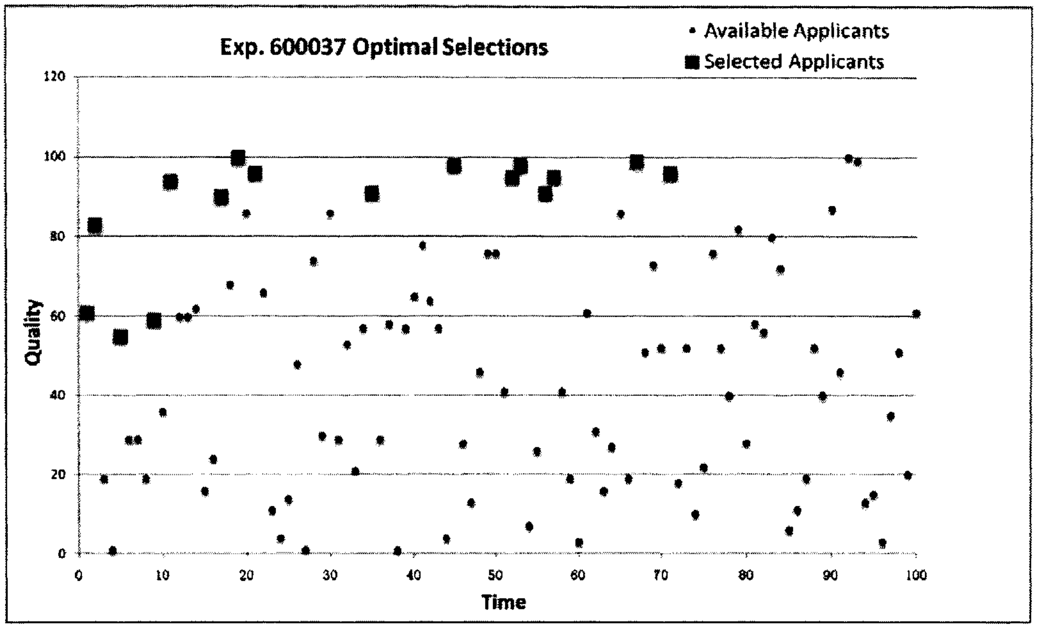

Figure 32. Selections Based on the Optimal Selection Strategy, Experiment ID: 600037

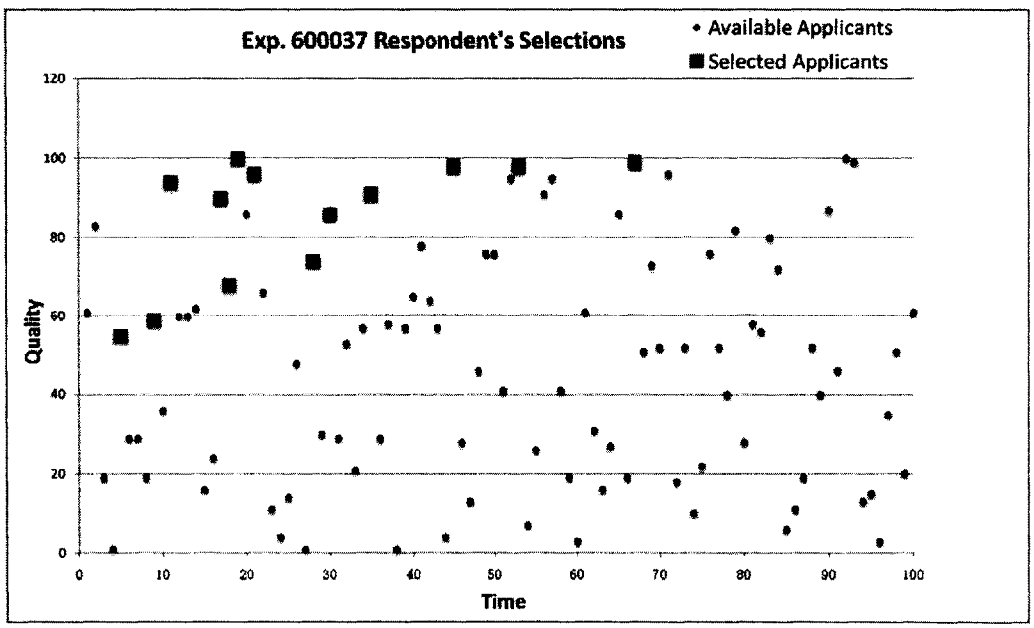

Figure 33. Respondent's Selections, Experiment ID: 600037

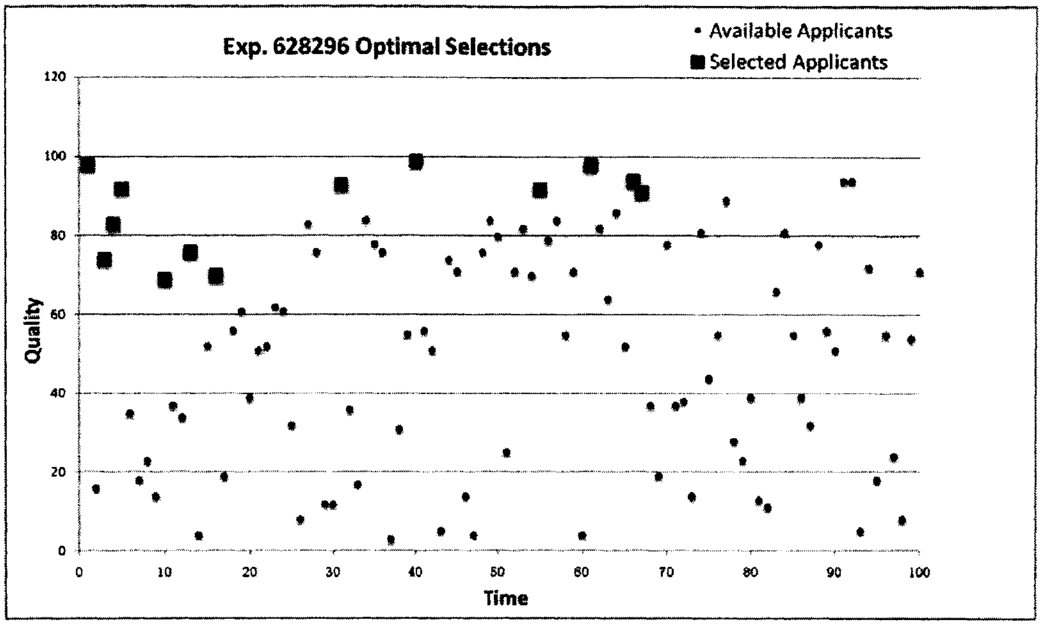

Figure 34. Selections Based on the Optimal Selection Strategy, Experiment ID: 628296

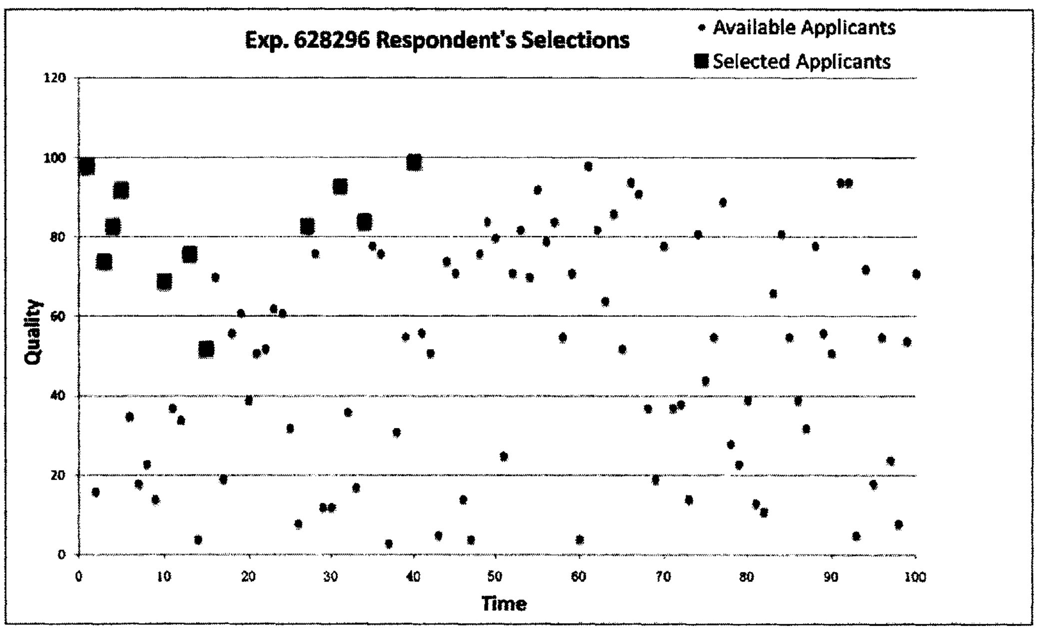

Figure 35. Respondent's Selections, Experiment ID: 628296

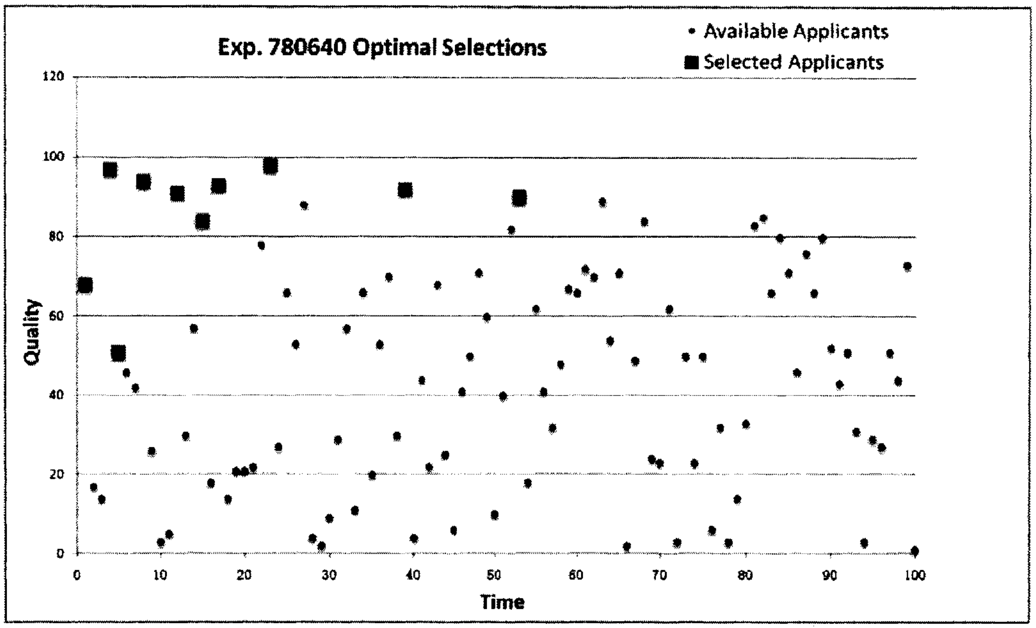

Figure 36. Selections Based on the Optimal Selection Strategy, Experiment ID: 780640

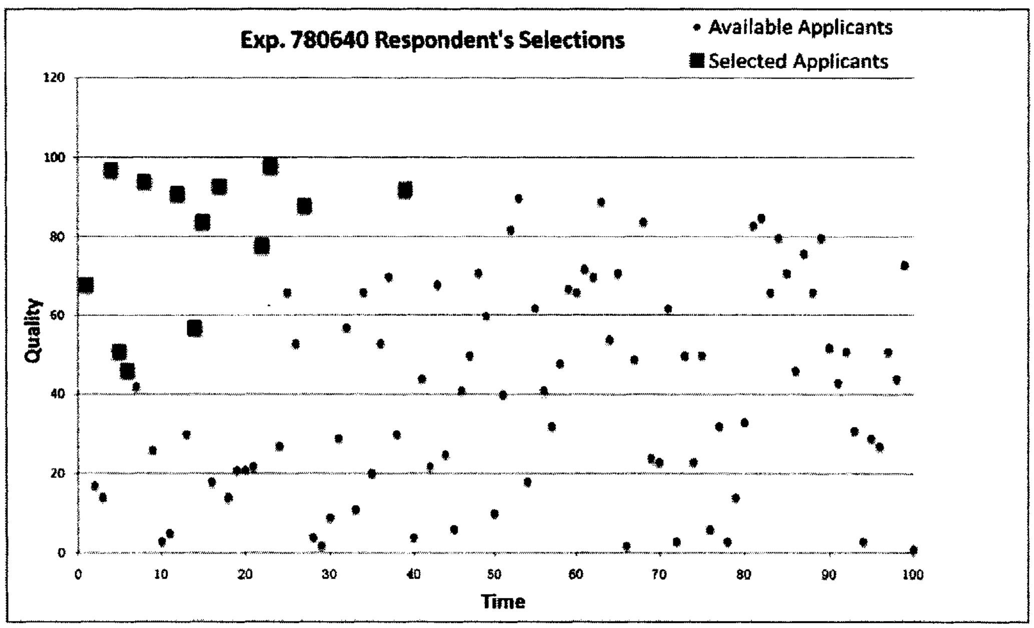

Figure 37. Respondent's Selections, Experiment ID: 780640

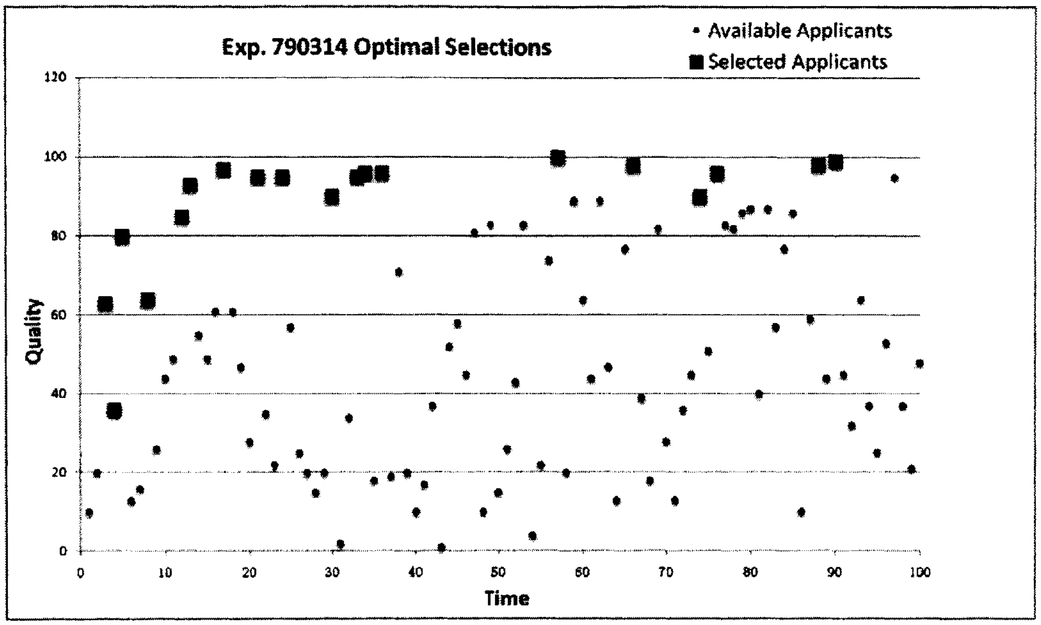

Figure 38. Selections Based on the Optimal Selection Strategy, Experiment ID: 790314

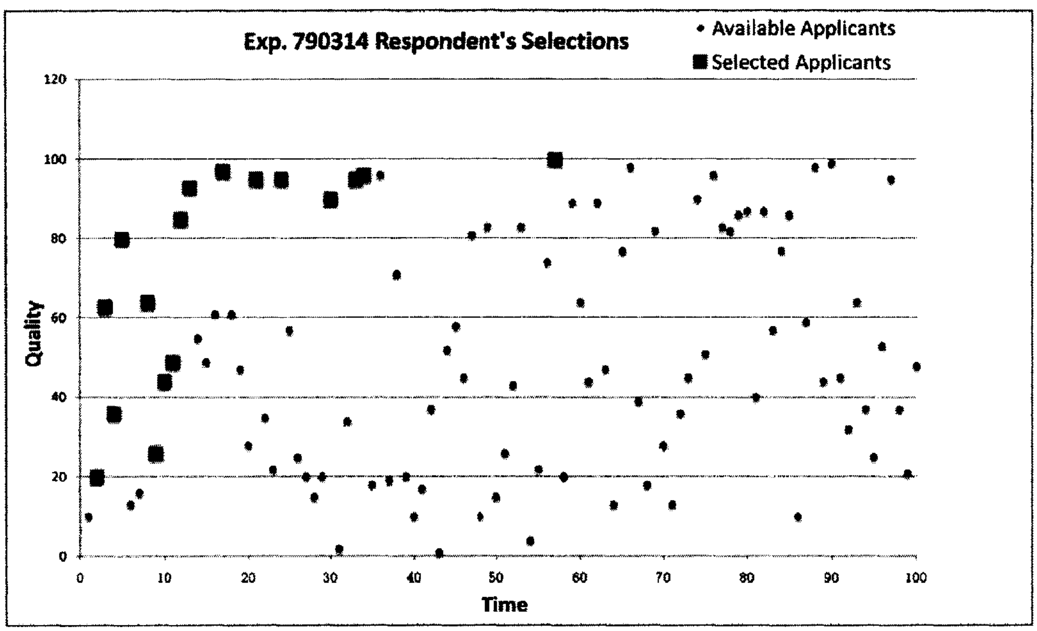

Figure 39. Respondent's Selections, Experiment ID: 790314

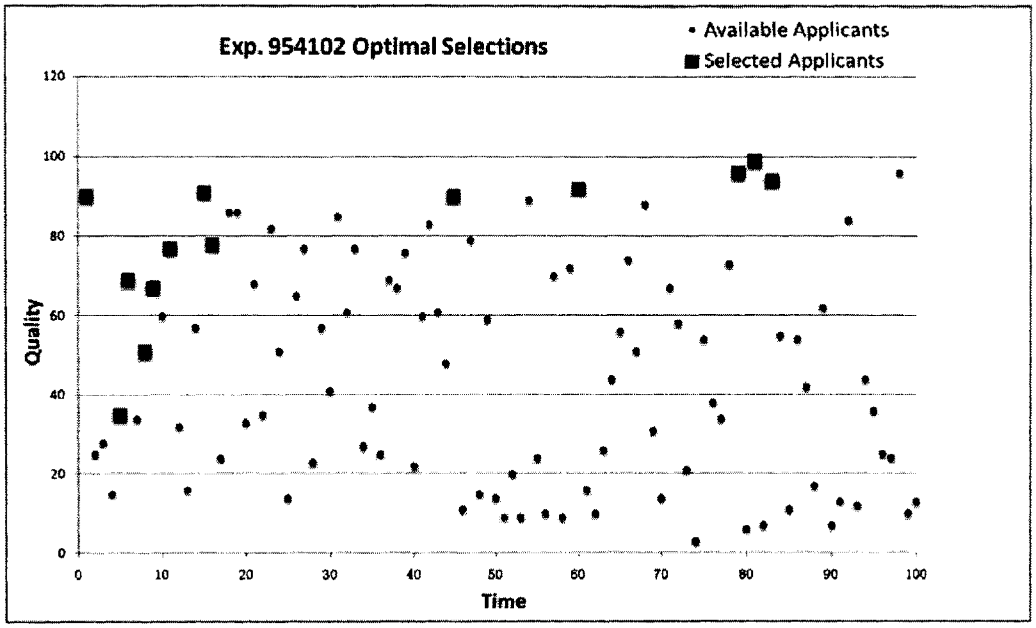

Figure 40. Selections Based on the Optimal Selection Strategy, Experiment ID: 954102

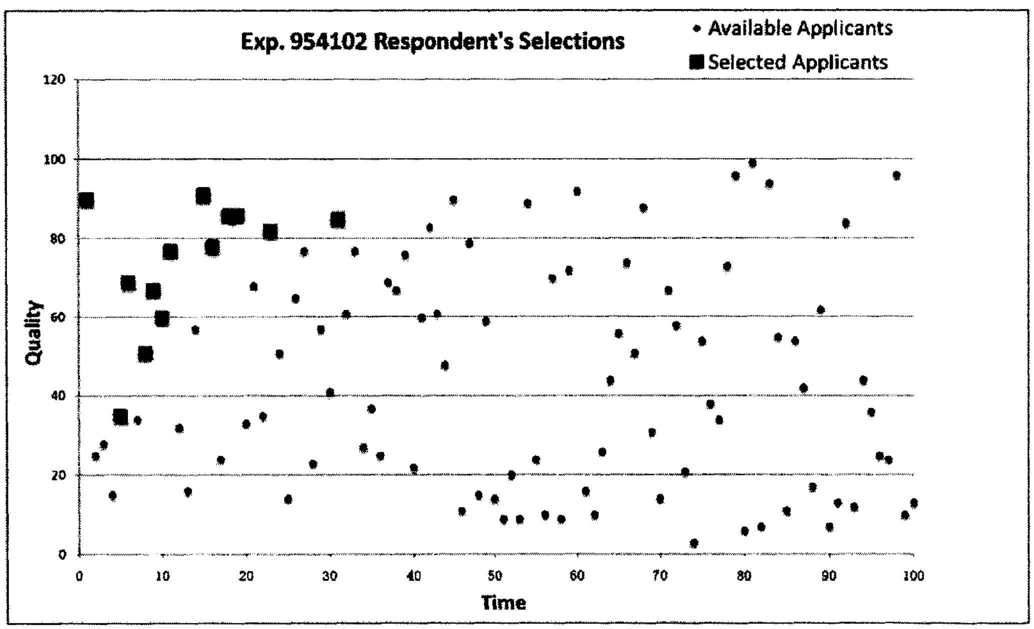

Figure 41. Respondent's Selections, Experiment ID: 954102

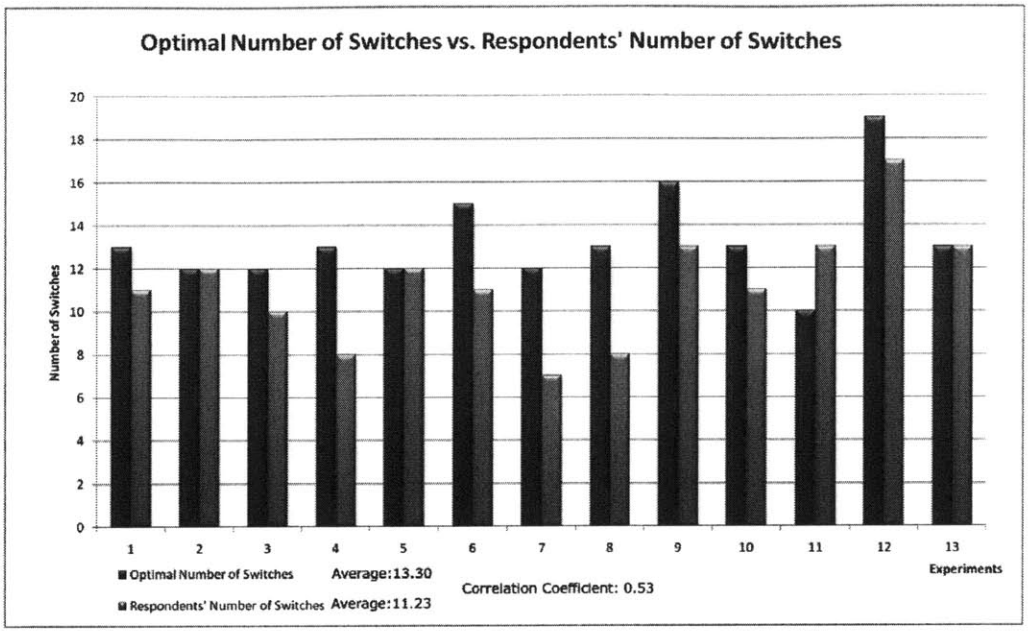

Figure 42. Optimal Number of Switches vs. Respondents' Number of Switches

Figure 42 compares the optimal number of switches and respondents' number of switches. The average number of switches based on the optimal selection strategy is 13.30, which is higher than that of respondents' number of switches 11.23. The correlation coefficient of these two variables is 0.53. As we can see from the plot, the optimal strategy generally leads to more switches and there is only one exception among the 13 experiments. This suggests that respondents might be less ambitious to get a higher payoff by frequently switching numbers in consideration of the switch costs.

Figure 43 shows the comparison between the approximate optimal payoffs that could be gained with the payoffs achieved by respondents in each experiment. Although we would expect that the optimal payoffs are always greater than respondents' payoffs, though there were a few respondents who actually achieved better results than the optimal strategy does. These exceptions are a consequence of the limited accuracy of our approximate optimal selection rule. The average optimal payoff is 52443, and the average of respondents' payoffs is 50296. The correlation coefficient of these two variables is 0.63.

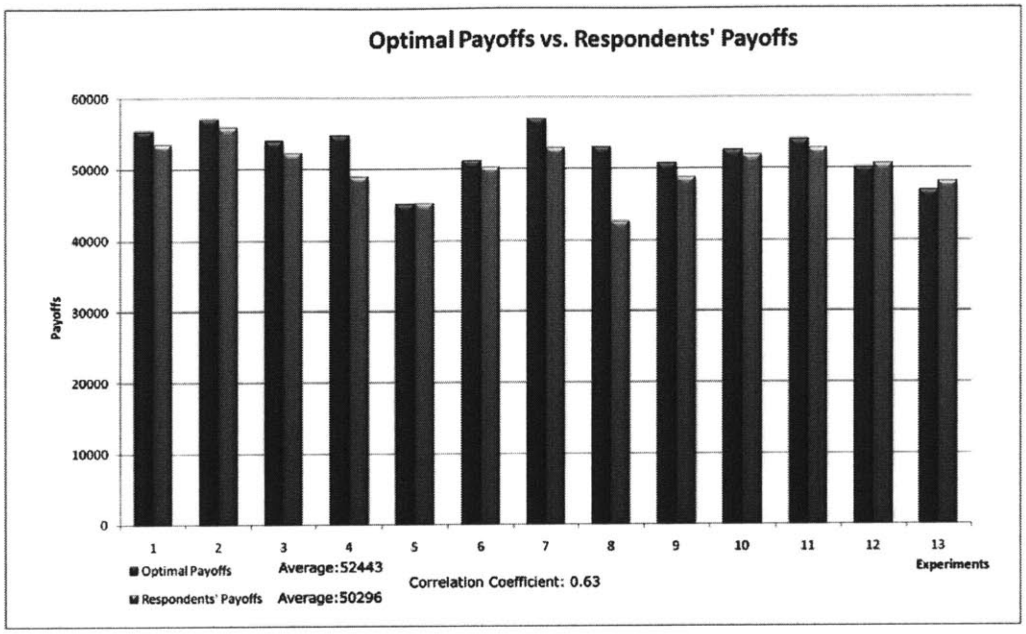

Figure 43. Optimal Payoffs vs. Respondents' Payoffs

Chapter 5. Summary and Future Directions

In this thesis, we have explored the optimal selection strategies for the multiple secretary problem with switch costs by using probabilistic reasoning and numerical analysis. We have examined the effect of various costs on optimal selection strategy in generalized secretary problems with no recall or replacement allowed. A single secretary problem with switch cost has been presented and its optimal selection rules in both infinite and finite time horizons have been given. The model is further extended to the case where we can hire more than one secretary at a time. The selection strategy for two secretary case has been determined and an approximate solution for the seven secretary case has been given. At the end of the thesis, experimental results have been compared against the optimal selection rule.

Numerous computational methods have been employed in this thesis. We have exploited numerical methods to calculate complicated integral equations. Dynamic programming and memoization method have been utilized to improve computation efficiency.

In the future, we can consider further improving our computation methodology to find out a more accurate optimal selection strategy for seven secretary problem with switch costs. The current method asks for a huge space requirement of around 32 exabits, which far exceeds the size of currently available drives.

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