SCALARIZATION IN SET OPTIMIZATION AND APPLICATIONS

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ABSTRACT

In this paper, we study the properties of the nonlinear scalarizing functions for set optimization problems and its applications. First, under suitable assumptions, we establish the semicontinuity and pseudo-monotonicity of such functions. Next, by using the above properties, we study the well-posedness of related equilibrium problems. Our results are new or improve the existing ones in the literature.

Keywords: Scalarizing function, set optimization problems, well-posedness, equilibrium problems.

1. INTRODUCTION

Scalarization is one of the most important techniques to study problems in optimization. There are two types of scalarizing functions, linear and nonlinear ones (see, for instance, [1-6] and the references therein). One of the most popular nonlinear scalarizing functions is the Gesterwizt's function. This function has been extended in several versions to deal with different problems. Motivated by the above observations, in this article, we propose a version of the scalarizing Gesterwitz's function in set optimization problems. We also establish the conditions for the semicontinuity and pseudo-monotonicity of this function. These properties can be applied to study the well-posedness of equilibrium problems.

The rest of the article is organized as follows. In Section 2, we state the set optimization problems and recall some preliminary results. Section 3 devotes to the properties of the nonlinear scalarizing function in set optimization problems. The applications of the above properties are presented in Section 4.

2. PRELIMINARIES

Let X, Y be two normed vector spaces. For $A\subset Y$, we denote by int$A$ and cl$A$ the interior and closure of A, respectively. Let C be a closed, convex and pointed cone in Y, with $\text{int}C\neq\emptyset$ and e is a fixed element of $\text{int}C$.

Let $\mathcal{P}_0(Y)$ be the family of all nonempty subsets of Y. For any $A,B\in\mathcal{P}_0(Y)$, we define the set less order relation $\lesssim^u$ and the strict one $<^u$ as follows

$$A\lesssim^u B\Leftrightarrow A\subset B-C,$$

$$A<^u B\Leftrightarrow A\subset B-\text{int}C.$$

It is said that $A \in \mathcal{P}_0(Y)$ is C-proper if $A+C \neq Y$, C-closed if $A+C$ is a closed set, C-bounded if for each neighborhood U of the origin in Y, there is some positive number t such that $A \subset tU + C$, C-compact if any cover of A of the form $\{U_\alpha + C : U_\alpha$ are open$\}$ admits a finite subcover.

From [7], every C-compact set is C-closed and C-bounded.

Definition 2.1. [4] Let $a \in Y$, the Gerstewizt's function $\phi_{e,a} : Y \to \mathbb{R}$ is defined by

$$\phi_{e,a}(y) = \min\{t \in \mathbb{R} : y \in te + a - C\}, \forall y \in Y.$$

The function $\phi_{e,A} : Y \to \mathbb{R} \cup \{-\infty\}$ is obtained by replacing a by $A \in \mathcal{P}_0(Y)$

$$\phi_{e,A}(y) = \inf\{t \in \mathbb{R} : y \in te + A - C\}, \forall y \in Y.$$

Clearly $\phi_{e,A}$ is continuous and it is easy to see that for any given $y \in Y$

$$\phi_{e,A}(y) = \inf_{a \in A}\{\phi_{e,a}(y)\}.$$

Let $\mathcal{P}_{0(-C)}(Y)$ be the family of all nonempty (-C)-proper subsets of Y.

Proposition 2.1. [4] *Let $A \in \mathcal{P}_{0(-C)}(Y)$ and $r \in \mathbb{R}$. Then for any $y \in Y$ the following statements are satisfied*

(i) $\phi_{e,A}(y) < r \Leftrightarrow y \in re + A - \mathrm{int}C$;

(ii) $\phi_{e,A}(y) \leq r \Leftrightarrow y \in re + cl(A - C)$;

(iii) $\phi_{e,A}(y) = r \Leftrightarrow y \in re + \partial(A - C)$, where ∂B is the topological boundary of *B*.

Let $F : X \rightrightarrows Y$ be a set-valued mapping and K is a nonempty subset of X. We consider the set optimization problem (SOP)

$$\min_{x \in K} F(x).$$

Definition 2.2. (See [8]) Let $x_0 \in K$, x_0 is said to be

(i) u-minimal solution of (SOP) if, for $x \in K, F(x) \lesssim^u F(x_0)$ implies $F(x_0) \lesssim^u F(x)$;

(ii) Weakly u-minimal solution of (SOP) if, for $x \in K, F(x) <^u F(x_0)$ implies $F(x_0) <^u F(x)$.

We denote the u-minimal solution set of (SOP) and the weakly u-minimal solution set of (SOP) by $E(F,K)$ and $W(F,K)$, respectively.

Remark 2.1. By Remark 2.1 of [8], $E(F,K) \subseteq W(F,K)$. Moreover, by Remark 2.5 of [8], if A is compact then $\mathrm{WMax}(A) \neq \emptyset$.

Definition 2.3. [9] A function $f : X \to \overline{\mathbb{R}}$ is said to be upper semicontinuous (u.s.c, shortly) at a point $x \in X$ if for every $\varepsilon>0$, there exists a neighborhood *U* of *x* such that $f(y) \leq f(x) + \varepsilon$ for all $y \in U$ when $f(x) > -\infty$, and $f(y) \to -\infty$ as $y \to x$ when $f(x) = -\infty$. *f* is said to be upper semicontinuous on *X* if it is upper semicontinuous at every point of *X*.

It follows from [9] that *f* is upper semicontinuous on *X* if and only if the set $\{x \in X \mid f(x) \geq r\}$ is closed for all $r \in \mathbb{R}$.

Definition 2.4. A set-valued mapping $G : X \rightrightarrows Y$ is said to be

(i) upper semicontinuous (usc, shortly) at $x_0 \in \operatorname{dom} G$ if for any open superset U of $G(x_0)$, there exists a neighborhood V of x_0 such that $G(V) \subset U$;

(ii) lower semicontinuous (lsc, shortly) at $x_0 \in \operatorname{dom} G$ if for any open subset U of Y with $G(x_0) \cap U \neq \emptyset$, there exists a neighborhood V of x_0 such that $G(x) \cap U \neq \emptyset, \forall x \in V$;

(iii) continuous at x_0 if it is both upper semicontinuous and lower semicontinuous at x_0.

Lemma 2.1. [8] *If K is nonempty compact and F is lsc on K with nonempty compact values, then $E(F,K) \neq \emptyset$.*

Let A be a nonempty subset of Y. $a \in A$ is said to be weak maximum point of A with respect to C, if $(A-a) \cap \operatorname{int} C = \emptyset$. We denote the set of all weak maximum point of A by WMax(A).

Lemma 2.2. [8] *For any $x_0 \in K$, if WMax($F(x_0)) \neq \emptyset$, then x_0 is a weakly u-minimal solution of (SOP) if and only if there does not exist $x \in K$ satisfying $F(x) <^u F(x_0)$.*

Lemma 2.3. [3] *Let $F: X \rightrightarrows Y$ be a set-valued mapping. Then the following assertions hold.*

(i) *F is usc at $\bar{x}$ and $F(\bar{x})$ is compact if and only if for any sequence $\{x_n\} \subset X$ with $x_n \to \bar{x}$ and $y_n \in F(x_n)$, there is a subsequence $\{y_{n_k}\}$ that converges to some $\bar{y} \in F(\bar{x})$.*

(ii) *F is lsc at $\bar{x}$ if and only if for any sequence $\{x_n\} \subset X$ with $x_n \to \bar{x}$ and $\bar{y} \in F(\bar{x})$, there exists a sequence $\{y_n\}$, $y_n \in F(x_n)$, such that $y_n \to \bar{y}$.*

3. PROPERTIES OF SCALARIZING FUNCTION

Proposition 3.1. *Assume that $A, B \in \mathcal{P}_{0(-C)}(Y)$ are (-C)-closed. Then*

(i) *$A \lesssim^u B$ if and only if $\phi_{e,A}(a) \geq \phi_{e,B}(a)$ for any $a \in A$.*

(ii) *$A <^u B$ if and only if $\phi_{e,A}(a) > \phi_{e,B}(a)$ for any $a \in A$.*

Proof. Since the proof techniques are similar, we only prove (i). The sufficient condition is followed by the definition of the function $\phi_{e,A}$. For the necessary condition, assume to the contrary that $A \not\subset B - C$. Then there is $a \in A$ such that $a \notin B - C$. On the one hand, by Proposition 2.1, $\phi_{e,B}(a) > 0$. On the other hand, by the definition of the function $\phi_{e,A}$, we have $\phi_{e,A}(a) \leq 0$, which is in contradiction with $\phi_{e,A}(a) \geq \phi_{e,B}(a)$. □

Inspired by the ideas in [4, 5], we propose the scalarizing function $G_e : \mathcal{P}_{0(-C)}(Y)^2 \to \mathbb{R} \cup \{\infty\}$ as follows.

$$G_e(A,B) = \sup_{b \in B}\{\phi_{e,A}(b)\}, \forall (A,B) \in \mathcal{P}_{0(-C)}(Y)^2.$$

Proposition 3.2. *Assume that $A, B \in \mathcal{P}_{0(-C)}(Y)$ are (-C)-compact sets. Then $G_e(A,B) < 0$ if and only if $B <^u A$.*

Proof. Suppose that $G_e(A,B)<0$. By the definition of G_e, for each $b\in B$, $\phi_{e,A}(b)<0$. By Proposition 2.1, $b\in A-\text{int}C$ and then $B<^u A$.

Conversely, assume that $B<^u A$. By Proposition 3.1, $\phi_{e,A}(b)<\phi_{e,B}(b),\forall b\in B$. Noting that $\phi_{e,B}(b)\le 0,\forall b\in B$, we obtain $\phi_{e,A}(b)<0,\forall b\in B$. We will show that $G_e(A,B)<0$.

Indeed, for each $b\in B$, since $\phi_{e,A}(b)<0$, there exists $r_b<0$ such that $b\in r_b e+A-C\subset r_b e+A-\frac{r_b}{2}e-\text{int}C-C$. Therefore,

$$B\subset\bigcup_{r_b}(r_b e+A-\frac{r_b}{2}e-\text{int}C-C)=\bigcup_{r_b}(\frac{r_b}{2}e+A-\text{int}C-C).$$

Noting that $\frac{r_b}{2}e+A-\text{int}C$ is open, from the (-C)-compactness of B, there exists $k\in\mathbb{N}$ such that

$$B\subset\bigcup_{i=1}^{k}(\frac{r_{b_i}}{2}e+A-\text{int}C-C)=\bigcup_{i=1}^{k}(\frac{r_{b_i}}{2}e+A-\text{int}C).$$

Let $\frac{r}{2}=\max_i\frac{r_{b_i}}{2}$, we have $B\subset\frac{r}{2}e+A-\text{int}C.$ By Proposition 2.1, $\phi_{e,A}(b)<\frac{r}{2}$. This implies that $G_e(A,B)<0.$ □

Proposition 3.3. Let $A,B\in\mathcal{P}_{0(-C)}(Y)$, s*uppose that* A *is* (-C)-*closed set, then* $G_e(A,B)\le r$ *if and only if* $B\subset re+A-C$.

Proof. $G_e(A,B)\le r$ if and only if $\phi_{e,A}(b)\le r,\forall b\in B$. The proof is completed by the (-C)-closedness of A and Proposition 2.1. □

Let X_1,X_2 be two normed vector spaces, $F_1:X_1\rightrightarrows Y$, $F_2:X_2\rightrightarrows Y$ be two set-valued mappings, $g(x,y):=G_e(F_1(x),F_2(y)),\forall(x,y)\in X_1\times X_2$.

We first establish the sufficient conditions for the upper semicontinuity of the mapping g as follows.

Theorem 3.1. *If* F_1 *is lsc with nonempty values,* F_2 *is usc with nonempty and compact values, then* g *is u.s.c on* $X_1\times X_2$.

Proof. For any $\alpha\in\mathbb{R}$, let

$$L:=\{(x,y)\in X_1\times X_2: g(x,y)\ge\alpha\}.$$

We will show that L is closed. Indeed, for any sequence $\{(x_n,y_n)\}\subseteq L$ with $(x_n,y_n)\to(x_0,y_0)$. If $(x_0,y_0)\notin L$ then

$$g(x_0,y_0):=G_e(F_1(x_0),F_2(y_0))=\sup_{b\in F_2(y_0)}\phi_{e,F_1(x_0)}(b)<\alpha.$$

This implies that

$$\phi_{e,F_1(x_0)}(b)<\alpha,\forall b\in F_2(y_0).$$

By Proposition 2.1,

$$F_2(y_0)\subset\alpha e+F_1(x_0)-\text{int}C. \tag{1}$$

We claim that $F_2(y_n) \subset \alpha e + F_1(x_n) - \text{int}C$ for n large enough. If not, there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ such that $F_2(y_{n_k}) \not\subset \alpha e + F_1(x_{n_k}) - \text{int}C$. Without loss of generality, we can assume that $F_2(y_n) \not\subset \alpha e + F_1(x_n) - \text{int}C, \forall n \in \mathbb{N}$. Hence, there exists $v_n \in F_2(y_n)$ such that

$$v_n \notin \alpha e + F_1(x_n) - \text{int}C, \forall n \in \mathbb{N}. \tag{2}$$

Since F_2 is usc with compact values, by Lemma 2.3, without loss of generality, there exists $v_0 \in F_2(y_0)$ such that $v_n \to v_0$. From (1), there exists $u_0 \in F_1(x_0)$ such that

$$v_0 - u_0 \in \alpha e - \text{int}C. \tag{3}$$

Since F_1 is lsc, by Lemma 2.3, there exists $u_n \in F_1(x_n)$ such that $u_n \to u_0$. Thus, (3) yields that $v_n - u_n \in \alpha e - \text{int}C$ for n large enough, which contradicts with (2). Thus, $F_2(y_n) \subset \alpha e + F_1(x_n) - \text{int}C$ for n large enough. Hence, $g(x_n, y_n) := G_e(F_1(x_n), F_2(y_n)) = \sup_{b \in F_2(y_n)} \phi_{e, F_1(x_n)}(b) < \alpha$, which contradicts with $\{(x_n, y_n)\} \subseteq L$. Therefore, $(x_0, y_0) \in L$ and g is u.s.c. □

We are now in the position to discuss the pseudo-monotonicity of the scalarizing function.

Definition 3.1. [10] The function $f : X \times X \to \mathbb{R}$ is said to be pseudo-monotone on $X \times X$ iff for any $x, y \in X$ with $x \neq y$,

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0.$$

For each $y \in X$, let

$$P(y) = \{x \in K : F(y) \not\leq^u F(x)\},$$

$$Q(y) = \{x \in K : F(x) \lesssim^u F(y)\}.$$

Theorem 3.2. *If for each $y \in X$, $P(y) \subset Q(y)$ and $F(x)$ is (-C)-compact for all $x \in X$, then $G_e(F(\cdot), F(\cdot))$ is pseudo-monotone on $X \times X$.*

Proof. For any $x, y \in X$ with $x \neq y$, suppose that $G_e(F(x), F(y)) \geq 0$. By Proposition 3.2, $F(y) \not\leq^u F(x)$. This means that $x \in P(y)$ and then $x \in Q(y)$, i.e., $F(x) \subset F(y) - C$. Noting that $F(y)$ is (-C)-closed, by Proposition 3.3, $G_e(F(y), F(x)) \leq 0$. The proof is completed. □

4. APPLICATIONS TO WELL-POSEDNESS OF EQUILIBRIUM PROBLEMS

Let $f : X \times X \to \mathbb{R}$, we consider the scalar equilibrium problem as follows

$(EP(f, K))$ Find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \forall y \in K.$$

We denote the solution set of $(EP(f, K))$ is $S(f, K)$.

Pick up the idea from [11], we propose the following assumptions

(A1) $h : X \times X \to \mathbb{R}$ is upper semicontinuous and pseudo-monotone on $X \times X$;

(A2) $\sup_{(x,y)\in X\times X} |h(x,y)| < +\infty$;

(A3) A is a nonempty compact subset of X and there exists $x\in A$ such that $h(x,y)\geq 0, \forall y\in A$.

Given $x\in X$, and two nonempty subsets A and B of X, we define

$$d(x,A):=\inf_{a\in A}\|x-a\|, \text{ and } e(A,B):=\sup_{a\in A} d(a,B),$$

and the Hausdorff distance between A and B

$$H(A,B)=\max\{e(A,B),e(B,A)\}.$$

Let $M=\{(h,A): h,A \text{ satisfies (A1),(A2),(A3)}\}$. According to the result in [10], M is a complete metric space with the metric ρ is defined as follows: for any $u_1=(h_1,A_1),\ u_2=(h_2,A_2)\in M$,

$$\rho(u_1,u_2):=\sup_{(x,y)\in X\times X}\|h_1(x,y)-h_2(x,y)\|+H(A_1,A_2).$$

For the sequence $\{u_n\}\subset M$, $u_n\to a$ if $\rho(u_n,a)\to 0$.

Definition 4.2. [11] *Let $(f,K)\in M$, the problem $(EP(f,K))$ associated with (f,K) is said to be generalized Hadamard well-posed iff for any $(f_n,K_n)\in M$ and $x_n\in S(f_n,K_n)$, $(f_n,K_n)\to(f,K)$ implies that $\{x_n\}$ has a subsequence converging to an element of $S(f,K)$.*

Lemma 4.1. [11] *For each $(f,K)\in M$, the problem $(EP(f,K))$ associated with (f,K) is generalized Hadamard well-posed.*

By employing the properties of G_e which is investigated above, we establish the sufficient condition for the well-posedness of the related equilibrium problem as follows.

Theorem 4.1. *If K is compact, F is continuous with nonempty, compact values and $P(y)\subset Q(y), \forall y\in X$, then the problem $(EP(G_e(F(\cdot),F(\cdot)),K))$ associated with $(G_e(F(\cdot),F(\cdot)),K)$ is generalized Hadamard well-posed.*

Proof. We will show that $(G_e(F(\cdot),F(\cdot)),K)\in M$. By Theorem 3.1, $G_e(F(\cdot),F(\cdot))$ is u.s.c on $X\times X$. By Theorem 3.2, $G_e(F(\cdot),F(\cdot))$ is pseudo-monotone on $X\times X$. Since F is compact-valued, it is easy to see that $\sup_{(x,y)\in X\times X}|G_e(F(x),F(y))|<+\infty$. By Lemma 2.1, $E(F,K)\neq\emptyset$. Moreover, $E(F,K)\subseteq W(F,K)$, and then by Remark 2.1, $W(F,K)$ is also nonempty.

Next, we claim that there exists $x\in K$ such that $G_e(F(x),F(y))\geq 0, \forall y\in K$. Suppose by contradiction that for each $x_0\in K$, there exists $y_0\in K$ such that $G_e(F(x_0),F(y_0))<0$. Then by Proposition 3.2,

$$F(y_0)<^u F(x_0). \tag{6}$$

Since F is compact-valued, by Remark 2.1 $\mathrm{WMax}(F(x_0))\neq\emptyset$. By choosing $x_0\in E(F,K)$, from Lemma 2.2, there does not exist $x\in K$ satisfying $F(x)<^u F(x_0)$. This contradicts with (6). Hence, there exists $x\in K$ such that $G_e(F(x),F(y))\geq 0, \forall y\in K$ and $(G_e(F(\cdot),F(\cdot)),K)\in M$. By Lemma 4.1, $EP(G_e(F(\cdot),F(\cdot)),K))$ associated with $(G_e(F(\cdot),F(\cdot)),K)$ is generalized Hadamard well-posed. □

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TÓM TẮT

VÔ HƯỚNG HÓA TRONG TỐI ƯU TẬP VÀ ỨNG DỤNG

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Trong bài viết này, nhóm tác giả nghiên cứu các tính chất của hàm vô hướng hóa phi tuyến cho bài toán tối ưu tập và ứng dụng của nó. Trước hết, dưới các giả thiết thích hợp, chúng tôi thiết lập tính nửa liên tục và tính giả đơn điệu của hàm này. Sau đó, bằng cách sử dụng các tính chất trên, chúng tôi nghiên cứu tính đặt chỉnh của bài toán cân bằng tương ứng. Các kết quả của chúng tôi là mới hoặc cải tiến các kết quả đã có.

Từ khóa: Hàm vô hướng hóa, bài toán tối ưu tập, đặt chỉnh, bài toán cân bằng.