AUTOMORPHISM-COINVARIANT MODULES OVER GENERAL COVERS

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ABSTRACT

In this paper, we recall concepts of dual automorphism invariant modules, $\chi-$cover and $\chi-$automorphism-coinvariant modules. Simultaneously, some of their basic properties are discussed. Moreover, it is shown that: If M is a right $R-$module with an epimorphic $\chi-$cover $p: X \to M$, then M is $\chi-$automorphism-coinvariant if and only if $g\left(\ker(p)\right) = \ker(p)$ for any automorphism g of X.

Keywords: Dual automorphism-invariant module, $\chi-$automorphism-coinvariant module, projective covers, general covers.

1. INTRODUCTION

Singh and Srivastava in [1] introduced the class of dual automorphism invariant modules, which is the duality concept of the automorphism invariant modules studied by Lee and Zhou in [2]. In [3], Guil Asensio, Tutuncu and Srivastava introduced the concepts of $\chi-$cover and $\chi-$automorphism-coinvariant module. Especially, if χ is the class of all projective right $R-$modules, then the $\chi-$cover of a module coincides with its projective cover. If χ is the class of all projective modules over a right perfect ring, $\chi-$automorphism-coinvariant modules are precisely dual automorphism invariant modules. The purpose of the article is to review recent results to prepare for the research of us in future. Throughout this article, all rings are associative rings with identity and all modules are right unital. For a submodule N of M, denote $N \leq M$ $\left(N < M\right)$ to mean that N is a submodule of M (respectively, proper submodule) and End(M) (Aut(M)) to indicate the endomorphism ring of M (respectively, automorphism). A submodule N of module M is called small in M (denoted as $N = M$) if $N + K \neq M$ for any proper submodule K of M.

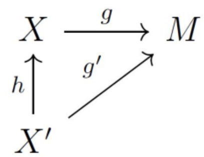
2. MAIN CONTENT

Recall that, let R be a ring and M be a right $R-$module. A projective cover of M is a pair (P, p) where P is a projective right $R-$module and $p: P \to M$ is a small epimorphism, e.i., p is a epimorphism and $\ker(p) = P$. It is well-known that if the existence of projective covers, then they are unique and up to isomorphisms in the following sense: If (Q, q) and (P, p) are two projective covers of M, then there exists an isomorphism h such that $p \circ h = q$. In [3], Guil Asensio, Tutuncu and Srivastava generalized the concept of projective cover as follows:

Let χ be a class of right $R-$modules. χ is said to be closed under isomorphisms, if $M \in \chi$ and $N \cong M$, then $N \in \chi$.

Definition 2.1. Let χ be a class of right R-modules whichbeclosed under isomorphisms. A homomorphism $g: X \to M$ of right $R-$modules is an $\chi-$cover of a module M provided that

(1) $X \in \chi$; and, for every homomorphism $g': X' \to M$ with $X' \in \chi$, there exists a homomorphism $h: X' \to X$ such that $g' = g \circ h$:

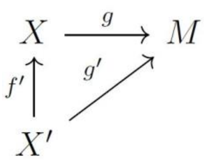


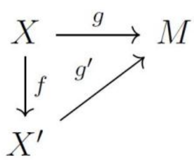
(2) $g = g \circ h$ implies that h is an automorphism for every endomorphism $h: X \to X$.

From above definition, it's easy to deduce some properties related to $\chi-$cover.

As same as uniqueness of projective cover up to isomorphism, we have the following remark.

Remark 2.2. If $g: X \to M$ and $g': X' \to M$ are two different $\chi-$covers of $R-$module M, then $X' \cong X$. Indeed, since both X and X' are $\chi-$covers of M, there exist homomorphisms $f': X' \to X$ and $f: X \to X'$ such that the following diagrams are commutative:





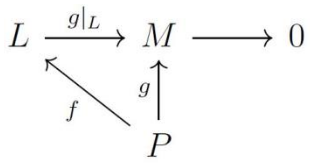
that is, $g' = g \circ f'$ and $g = g' \circ f$ which imply that $g = g \circ f' \circ f$ and $g' = g' \circ f \circ f'$. By the hypothesis (2) in the definition, both $f \circ f'$ and $f' \circ f$ are automorphisms. This implies that both f and f' are isomorphisms.

Theorem 2.3. [4,Theorem 1.2.10] Let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$, where $M_i (i = 1,...,n)$ are submodules of M, $g_i : X_i \to M_i$ are $\chi-$covers of M_i. Then, $\oplus g_i : \oplus X_i \to M$ is a $\chi-$cover of M.

In case χ is the class of all projective right R-modules, then $\chi-$cover of a module coincides with its projective cover. In other words, we have the following theorem:

Theorem 2.4. [4, Theorem 1.2.12] If χ be a class of all projective right $R-$modules then the homomorphism $g: P \to M$ is a $\chi-$cover of right $R-$module M if and only if it is a projective cover of M.

Proof. $(\Rightarrow)$ Let $g: P \to M$ be a $\chi-$cover of M. As every module M is an epimorphic image of a projective module, then, let $g': P' \to M, P' \in \chi$ be an epimorphism. By the condition (1) of $\chi-$cover, there exists $h: P' \to P$ such that $g' = g \circ h.$ Since g' is an epimorphism, it follows that g is an epimorphism. Take L is a submodule of P such that $L + \ker(g) = P,$ then $g|_L: P \to M$ is an epimorphism. Since, $P \in \chi$ is a projective module, there exists $f: P \to L$ such that $g = g|_L \circ f$



Moreover, $g|_L = g \circ i$ where $i: L \to P$ is a canonical injection. Therefore, $g = g \circ i \circ f = g \circ f.$ By the condition (2) of $\chi-$cover, f must be an automorphism of P. It follows that $L = P$. Thus, $\ker(g) = P$. Thus $g: P \to M$ is a projective cover of M.

$(\Leftarrow)$ Assume $g: P \to M$ is a projective cover of M, clearly g satisfy the condition (1) of $\chi-$cover. Let f be a endomorphism of P satisfying $g = g \circ f$, then for all $x \in P$,

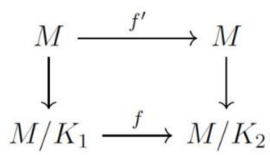
$$g(x) = g \circ f(x) \Leftrightarrow g\left(x - f(x)\right) = 0.$$

This means $x - f(x) = x' \in \ker(g).$ It implies that $x = x' + f(x), \forall x \in P.$ Thus $P = \ker(g) + f(P).$ As $\ker(g) = P$, then $P = f(P)$. It follows that f is an epimorphism. On the other hand, since P is projective and $f: P \to P$ is an epimorphism. Therefore, exists $h: P \to P$ such that $f \circ h = 1_P$, then h is monomorphic. For all $x \in P$,

$$f(x) = f \circ (h \circ f)(x) \Leftrightarrow f\left(x - h \circ f(x)\right) = 0.$$

This shows that $x - h \circ f(x) = x' \in \ker(f).$ Thus $x = h \circ f(x) + x', \forall x \in P,$ so $P = h(P) + \ker(f).$ Since $g = g \circ f$, $\ker(f) \leq \ker(g) = P$. It follows that $P = h(P)$, i.e., h is epimorphic. Hence h is a automorphism, so f is a automorphism. Thus $g: P \to M$ is a $\chi-$cover of $M.$

In 2012, Singh and Srivastava in [1] introduced the class of dual automorphism invariant module, which is the duality concept of the automorphism invariant module studied by Lee and Zhou in [2]. A right $R-$module M is called a dual automorphism-invariant module if whenever K_1 and K_2 are small submodules of M, then any epimorphism $f: M/K_1 \to M/K_2$ with small kernel lifts to an endomorphism f' of $M.$



These authors also showed that, the above endomorphism f' is a isomorphism and various properties of dual automorphism invariant modules have been studied. In 2015, Guil Asensio, Tutuncu and Srivastava introduced the concept of $\chi-$automorphism-coinvariant module in [3]. $\chi-$automorphism-coinvariant module can be considered the general concept of dual automorphism invariant module.

Definition 2.5. A right $R-$module M having a $\chi-$cover $p: X \to M$ is said to be $\chi-$automorphism-coinvariant if for any automorphism g of X, there exists an endomorphism f of M such that $f \circ p = p \circ g$:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & M \end{array}$$

Remark 2.6. A $\chi-$cover $p: X \to M$ of M is said to be epimorphic if p is an epimorphism. Let $\chi-$automorphism-coinvariant module M with p is an epimorphic $\chi-$cover. Since g is an automorphism of X, then g^{-1} also is an automorphism of X. There exists an endomorphism f' of M such that $f' \circ p = p \circ g^{-1}$. It follow that $p = f \circ f' \circ p$. Since p is epimorphic, $f \circ f' = id_M$. By a similar argument as above, $f' \circ f = id_M$. Thus $f \in \mathrm{Aut}(M)$. Consider

$\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(M)$,

g a f satisfy $f \circ p = p \circ g$. Clear, φ is a group epimorphism with $\ker(\varphi)$ is the set of the automorphisms g of X such that $p \circ g = p$. Therefore, if we consider a module M with an epimorphic $\chi-$cover, then M is $\chi-$automorphism-coinvariant if and only if p induces a group isomorphism

$\bar{\varphi} : \mathrm{Aut}(M) \cong \mathrm{Aut}(X)/\ker(\varphi)$.

We also denote $\ker(\varphi)$ is coGal(X) and call coGal(X) the coGalois group of the cover p.

Theorem 2.7. [3, Lemma 4.8] If M is $\chi-$automorphism-coinvariant and every direct summand of M has a $\chi-$cover, then any direct summand of M is $\chi-$automorphism-coinvariant.

Theorem 2.8. [1,Theorem 27] Let P be a projective module and $K = P$.Then $M = K/P$ is dual automorphism-invariant if and only if $\sigma(K) = K$ for any automorphism σ of P.

We now generalize the Theorem 2.8 as follows.

Theorem 2.9. Let M be a right $R-$module with an epimorphic $\chi-$cover $p: X \to M$. Then M is $\chi-$automorphism-coinvariant if and only if $g(\ker(p)) = \ker(p)$ for any automorphism g of X.

Proof. $(\Rightarrow)$ Assume M is $\chi-$automorphism-coinvariant. Then, for any automorphism g of X, there exists an endomorphism f of M such that $p \circ g = f \circ p$. Hence,

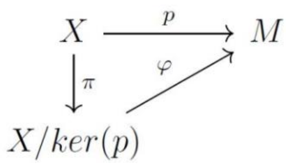
$$p \circ g\left(\ker(p)\right) = f \circ p\left(\ker(p)\right) = 0$$

so $g\left(\ker(p)\right) \le \ker(p)$. Since g^{-1} is automorphism of X, $g^{-1}\left(\ker(p)\right) \le \ker(p)$, it follows that $\ker(p) \le g\left(\ker(p)\right)$. Thus, $g\left(\ker(p)\right) = \ker(p)$.

$(\Leftarrow)$ Consider the following homomorphism:

$\varphi : X / \ker(p) \to M,\ \varphi\left(x + \ker(p)\right) = p(x).$

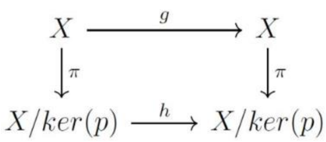
It easily to see that φ is well-defined and $\varphi \circ \pi = p$, where π is a canonical projection,



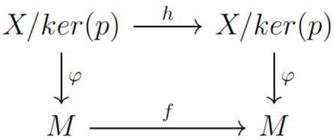
Since p is an epimorphism, φ is epimorphic. Moreover, φ is monomorphic, so φ is an isomorphism. Let g be an arbitrary automorphism of X. Consider

$h : X / \ker(p) \to X / \ker(p), h(x + \ker(p)) = g(x) + \ker(p).$

By $g\left(\ker(p)\right) = \ker(p)$, then h is well-defined. Clearly, we have $\pi \circ g = h \circ \pi$



Since $\varphi : X / \ker(p) \to M$ is an isomorphism, take $f = \varphi \circ h \circ \varphi^{-1} \in \mathrm{End}(M)$. Then the following diagram is commutative:



that is, $\varphi \circ h = f \circ \varphi.$ Hence

$p \circ g = \varphi \circ \pi \circ g = \varphi \circ h \circ \pi = f \circ \varphi \circ \pi = f \circ p.$

This proves that M is $\chi-$ automorphism-coinvariant.

Recall, a right perfect ring is a type of ring in which all right modules have projective covers. Let χ be a class of all projective right modules over the right perfect ring R, then we have the following corollary:

Corollary 2.10. For a right $R-$module M, the following statements are equivalent:

(1) M is a dual automorphism-invariant module;

(2) M is a $\chi-$ automorphism-coinvariant module.

Proof. It is straightforward from Theorem 2.8 and Theorem 2.9.

3. CONCLUSION

We have just discussed concepts of $\chi-$ cover and and $\chi-$ automorphism-coinvariant modules and obtained some basic properties of them. Furthermore, we will study $\chi-$ automorphism-coinvariant associated with conditions as regularity, non-singularity, exchange property, finitely generated, etc. and hope to get some interesting results. We guarantee that the results in the paper belong to us and are completely different from existing ones.

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TÓM TẮT

MÔ ĐUN ĐỐI BẤT BIẾN ĐẲNG CẤU TRÊN PHỦ TỔNG QUÁT

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Trong bài báo này, chúng tôi nhắc lại các khái niệm về mô đun đối bất biến đẳng cấu, phủ tổng quát, mô đun $\chi -$ đối bất biến đẳng cấu. Đồng thời, một số tính chất cơ bản của chúng cũng được thảo luận đến. Ngoài ra, chúng tôi chứng minh rằng: Cho M là một $R-$ mô đun phải với một toàn cấu phủ tổng quát $p: X \to M$, khi đó M là $\chi -$ đối bất biến đẳng cấu khi và chỉ khi $g\left(\ker(p)\right) = \ker(p)$ với mọi tự đẳng cấu g của X.

Từ khóa: Mô đun đối bất biến đẳng cấu, mô đun $\chi -$ đối bất biến đẳng cấu, phủ xạ ảnh, phủ tổng quát.