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# Weak two-scale convergence in $L^2$ for a two-dimensional case

## Hội tụ hai-kích thước yếu trong $L^2$ cho một trường hợp hai chiều

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**Abstract**

In this paper, we present definitions and some properties of the weak two-scale convergence (introduced by Nguetseng in 1989) for component-wise vector or matrix functions within a two-dimensional case.

*Keywords:* two-scale homogenization; weak two-scale convergence; two-dimensional

**Tóm tắt**

Trong bài báo này, chúng tôi trình bày các định nghĩa và một số tính chất của hội tụ hai-kích thước yếu (được giới thiệu bởi Nguetseng vào năm 1989) cho các hàm vectơ hoặc ma trận trong một trường hợp hai chiều.

*Từ khóa:* đồng nhất hóa hai-kích thước; hội tụ hai-kích thước yếu; hai chiều

## 1. Introduction

Let us consider in dimension two, a bounded reference domain $\Omega = \Omega^1 \times \Omega^2 \in \mathbb{R} \times \mathbb{R}$ and a variable $\boldsymbol{x} = (x^1, x^2) \in \Omega$. Within two-scale homogenization theory, when it is not possible to calculate limit in terms of the usual weak limit, it can be possible in terms of two-scale limit introduced in 1989 by Nguetseng [1]. In this spirit, we first present a brief review of the usual weak convergence in $\boldsymbol{L}^2(\Omega)$ then the definitions and properties of the weak two-scale convergence for component-wise vector or matrix functions [2, 3, 4, 5], in a two-dimensional case.

## 2. Preliminaries

Latin indices vary in the set $\{1, 2\}$. The space of functions, vector fields in $\mathbb{R}^2$, and $2 \times 2$ matrix fields, defined over $\Omega$ are respectively denoted by italic capitals (e.g. $L^2(\Omega)$), boldface Roman capitals (e.g. $\boldsymbol{V}$), and special Roman capitals (e.g. $\mathbb{S}$).

Throughout the study, we use the following list of notations [2]:

- $Y := [0, 1]^2$ is the reference periodic cell.

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- $\boldsymbol{C}_0(\Omega)$ is the space of functions that vanish at infinity.
- We denote by $\boldsymbol{C}^{\infty}_{\text{per}}(Y)$ the $Y$-periodic $\boldsymbol{C}^{\infty}$ vector-valued functions in $\mathbb{R}^2$. Here, $Y$-periodic means 1-periodic in each variable $y^i, i = 1,2$.
- $\boldsymbol{H}^1_{\text{per}}(Y)$, as the closure for the $\boldsymbol{H}^1$-norm of $\boldsymbol{C}^{\infty}_{\text{per}}(Y)$, is the space of vector-valued functions $\boldsymbol{v} \in \boldsymbol{L}^2(Y)$ such that $\boldsymbol{v}(y)$ is $Y$-periodic in $\mathbb{R}^2$.
- $$\langle \boldsymbol{v} \rangle_y = \frac{1}{|Y|} \int_Y \boldsymbol{v}(y)\,\mathrm{d}y.$$
- $$\boldsymbol{H}_{\text{per}}(Y) := \{\boldsymbol{v} \in \boldsymbol{H}^1_{\text{per}}(Y) \,|\, \langle \boldsymbol{v} \rangle_y = 0\}.$$
- We write $\cdot$ for the canonical inner products in $\mathbb{R}^2$ and $\mathbb{R}^{2\times 2}$, respectively.
- $\lesssim$ means $\leq$ up to a multiplicative constant that only depends on $\Omega$ when appropriate.

The Sobolev norm $\|\cdot\|_{\boldsymbol{W}^{1,2}_0(\Omega)}$ is of the form

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_0(\Omega)} = (\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)})^{\frac{1}{2}};$$

here, $\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} := \||\boldsymbol{v}|\|_{L^2(\Omega)}$, where $|\boldsymbol{v}|$ denotes the Euclidean norm of the 2-component vector-valued function $\boldsymbol{v}$, and $\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2(\Omega)} := \||\nabla \boldsymbol{v}|\|_{L^2(\Omega)}$, where $|\nabla \boldsymbol{v}|$ denotes the Frobenius norm of the $2 \times 2$ matrix $\nabla \boldsymbol{v}$. We recall that the Frobenius norm on $\mathbb{L}^2(\Omega)$ is defined by $|\boldsymbol{X}|^2 := \boldsymbol{X} \cdot \boldsymbol{X} = \text{tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})$.

Let $\epsilon$ be a natural small scale. For prospective applications in homogenization, based on [6, 7, 8, 9], we consider $\boldsymbol{u}_\epsilon(\boldsymbol{x}) \in \boldsymbol{W}^{1,2}_0(\Omega)$ depending only on $x^1$, that is, $\boldsymbol{u}_\epsilon(\boldsymbol{x}) = \boldsymbol{u}_\epsilon(x^1)$, with boundary conditions of Neumann type. As noticed in [10], we do not discriminate a function on $\mathbb{R}$ from its extension to $\mathbb{R}^2$ as a function of the first variable only. We assume that $\boldsymbol{u}_\epsilon(x^1) = \boldsymbol{u}\left(\frac{x^1}{\epsilon}\right)$ is a periodic function in $x^1$ with period $\epsilon$, equivalently, $\boldsymbol{u}\left(\frac{x^1}{\epsilon}\right) = \boldsymbol{u}(y^1)$ is a periodic function in $y^1$ with period 1. It means that for any integer $k$,

$$\boldsymbol{u}_\epsilon(x^1) = \boldsymbol{u}_\epsilon(x^1 + \epsilon) = \boldsymbol{u}_\epsilon(x^1 + k\epsilon),$$

equivalently,

$$\boldsymbol{u}\left(\frac{x^1}{\epsilon}\right) = \boldsymbol{u}\left(\frac{x^1}{\epsilon} + 1\right) = \boldsymbol{u}\left(\frac{x^1}{\epsilon} + k1\right) = \boldsymbol{u}(y^1 + k).$$

## 3. Weak convergence

We describe the basic notions of the theory of two-scale convergence (thanks to [4, 5]). Two-scale convergence here can be viewed as a generalized version of the usual weak convergence in the Hilbert space $\boldsymbol{L}^2(\Omega)$, which is defined as follows [4].

Let us consider a sequence of functions $\boldsymbol{u}_\epsilon \in \boldsymbol{L}^2(\Omega)$. By definition, $(\boldsymbol{u}_\epsilon)$ is bounded in $\boldsymbol{L}^2(\Omega)$ if

$$\limsup_{\epsilon \to 0} \int_\Omega |\boldsymbol{u}_\epsilon|^2\,\mathrm{d}x \leq c < \infty,$$

for some positive constant $c$.

We say that a sequence $(\boldsymbol{u}_\epsilon(\boldsymbol{x})) \in \boldsymbol{L}^2(\Omega)$ is weakly convergent to $\boldsymbol{u}(\boldsymbol{x}) \in \boldsymbol{L}^2(\Omega)$ as $\epsilon \to 0$, denoted by $\boldsymbol{u}_\epsilon \rightharpoonup \boldsymbol{u}$, if

$$\lim_{\epsilon \to 0} \int_\Omega \boldsymbol{u}_\epsilon(\boldsymbol{x}) \cdot \boldsymbol{\phi}\,\mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\phi}\,\mathrm{d}x, \tag{1}$$

for any test function $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega)$.

Moreover, a sequence $(\boldsymbol{u}_\epsilon)$ in $\boldsymbol{L}^2(\Omega)$ is defined to be strongly convergent to $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ as $\epsilon \to 0$, denoted by $\boldsymbol{u}_\epsilon \to \boldsymbol{u}$, if

$$\lim_{\epsilon \to 0} \int_\Omega \boldsymbol{u}_\epsilon \cdot \boldsymbol{v}_\epsilon\,\mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}x, \tag{2}$$

for every sequence $(\boldsymbol{v}_\epsilon) \in \boldsymbol{L}^2(\Omega)$ which is weakly convergent to $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$.

We then have the following well-known weak convergence properties in $\boldsymbol{L}^2(\Omega)$.

(a) Any weakly convergent sequence is bounded in $\boldsymbol{L}^2(\Omega)$.

(b) Compactness principle: any bounded sequence in $\boldsymbol{L}^2(\Omega)$ contains a weakly convergent subsequence.

(c) If a sequence $(\boldsymbol{u}_\epsilon)$ is bounded in $\boldsymbol{L}^2(\Omega)$ and (1) holds for all $\boldsymbol{\phi} \in \boldsymbol{C}_0^\infty(\Omega)$, then $\boldsymbol{u}_\epsilon \rightharpoonup \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$.

(d) If $\boldsymbol{u}_\epsilon \to \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{v}_\epsilon \rightharpoonup \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, then

$$\lim_{\epsilon\to 0}\int_\Omega \boldsymbol{u}_\epsilon \cdot \boldsymbol{v}_\epsilon \,\mathrm{d}x = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x.$$

(e) Weak convergence of $(\boldsymbol{u}_\epsilon)$ to $\boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$ together with

$$\lim_{\epsilon\to 0}\int_\Omega |\boldsymbol{u}_\epsilon|^2\,\mathrm{d}x = \int_\Omega |\boldsymbol{u}|^2\,\mathrm{d}x$$

is equivalent to strong convergence of $(\boldsymbol{u}_\epsilon)$ to $\boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$.

Throughout this paper, we denote by $Y = [0,1]^2$ the cell of periodicity. (In our case, a periodic cell has the form $Y = [0,1]\times[0,1]$.) The mean value of a 1-periodic function $\boldsymbol{\psi}(y^1)$ is denoted by $\langle\boldsymbol{\psi}\rangle$, that is,

$$\langle\boldsymbol{\psi}\rangle \equiv \int_{Y^1}\boldsymbol{\psi}(y^1)\,\mathrm{d}y^1.$$

Recall that $y^1 = \epsilon^{-1}x^1$, and we do not distinguish between a function on $Y^1$ and its extension to $Y$ as a function of the first variable only.

Also, here, the symbol $\boldsymbol{L}^2(Y)$ works not only for functions defined on $Y$ but also for the space of functions in $\boldsymbol{L}^2(Y)$ extended by 1-periodicity to the whole of $\mathbb{R}^2$. Similarly, $\boldsymbol{C}_{\text{per}}^\infty(Y)$ denotes the space of infinitely differentiable 1-periodic functions on the whole $\mathbb{R}^2$.

For later discussion, we introduce the following classical result.

**Lemma 3.1 (The mean value property).** *Let $\boldsymbol{h}(y^1)$ be a 1-periodic function on $\mathbb{R}$ and $\boldsymbol{h} \in \boldsymbol{L}^2(Y^1)$. Then, for any bounded domain $\Omega$, there holds the weak convergence*

$$\boldsymbol{h}\left(\frac{x^1}{\epsilon}\right) \rightharpoonup \langle\boldsymbol{h}\rangle \ in\ \boldsymbol{L}^2(\Omega) \ as\ \epsilon\to 0. \tag{3}$$

*Proof.* The proof is based on property (c) and can be found in [4]. □

## 4. Weak two-scale convergence

As mentioned in [4], in homogenization theory, one often has to handle quantities of the form (for our case)

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon(\boldsymbol{x})\left(\phi(\boldsymbol{x})h\left(\frac{x^1}{\epsilon}\right)\right)\mathrm{d}x,$$

where $u_\epsilon \rightharpoonup u, \phi \in C_0^\infty(\Omega)$ a scalar function, $h \in C_{\text{per}}^\infty(Y^1)$. In general, it is not possible to calculate this limit in terms of the usual weak limit $u$. However, it is possible in terms of the two-scale limit introduced in 1989 by Nguetseng [1]. In this spirit, we have the following definition of weak two-scale convergence in $L^2(\Omega)$ [2, 3].

**Definition 4.1.** *Let $(u_\epsilon)$ be a bounded sequence in $L^2(\Omega)$. If there exist a subsequence, still denoted by $u_\epsilon$, and a function $u(\boldsymbol{x}, y^1) \in L^2(\Omega\times Y^1)$, where $Y^1 = [0,1]$ such that*

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon(\boldsymbol{x})\left(\phi(\boldsymbol{x})h\left(\frac{x^1}{\epsilon}\right)\right)\mathrm{d}x = \int_{\Omega\times Y^1} u(\boldsymbol{x},y^1)(\phi(\boldsymbol{x})h(y^1))\,\mathrm{d}x\,\mathrm{d}y^1 \tag{4}$$

*for any $\phi \in C_0^\infty(\Omega)$ and any $h \in C_{per}^\infty(Y^1)$, then such a sequence $u_\epsilon$ is said to weakly two-scale converge to $u(\boldsymbol{x}, y^1)$. This convergence is denoted by $u_\epsilon(\boldsymbol{x}) \rightharpoonup\!\!\!\rightharpoonup u(\boldsymbol{x}, y^1)$.*

For vector (or matrix) $\boldsymbol{u}_\epsilon$, equation (4) implies

$$\lim_{\epsilon\to 0}\int_\Omega \boldsymbol{u}_\epsilon(\boldsymbol{x})\cdot\boldsymbol{\Phi}\left(\boldsymbol{x},\frac{x^1}{\epsilon}\right)\mathrm{d}x = \int_{\Omega\times Y^1}\boldsymbol{u}(\boldsymbol{x},y^1)\cdot\boldsymbol{\Phi}(\boldsymbol{x},y^1)\,\mathrm{d}x\,\mathrm{d}y^1, \tag{5}$$

for every $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\Omega;\boldsymbol{C}_{\text{per}}(Y^1))$, whose choice is explained in [11] (p. 8).

The Definition 4.1 makes sense because of the following compactness result, which was proved in [12] and first in [1].

**Theorem 4.2.** *Any bounded sequence $u_\epsilon \in \boldsymbol{L}^2(\Omega)$ contains a weakly two-scale convergent subsequence.*

*Proof.* The proof is obtained as in [4] or [5] with the help of the mean value property (3). □

**Remark 4.3.** *Regarding the class of test functions $\phi \in C_0^\infty(\Omega), h \in C_{per}^\infty(Y^1)$ in condition of (4), it can be extended (with the help of the density argument) to the class of test functions $\phi \in C_0^\infty(\Omega), h \in L^2(Y^1)$.*

Consequently, the convergence $u_\epsilon \rightharpoonup u$ implies the convergence

$$u_\epsilon(\boldsymbol{x}) b\left(\frac{x^1}{\epsilon}\right) \rightharpoonup u(\boldsymbol{x}, y^1) b(y^1), \quad \forall\, b \in L^\infty(Y^1). \tag{6}$$

We now have the following lower semicontinuity property [4].

**Lemma 4.4.** *If $u_\epsilon(\boldsymbol{x}) \rightharpoonup u(\boldsymbol{x}, y^1)$, then*

$$\liminf_{\epsilon\to 0} \int_\Omega |u_\epsilon(\boldsymbol{x})|^2 \,\mathrm{d}x \geq \int_{\Omega\times Y^1} |u(\boldsymbol{x}, y^1)|^2 \,\mathrm{d}x\,\mathrm{d}y^1. \tag{7}$$

*Proof.* The proof can be found in [4] or [5]. Specifically, denote by $\mathscr{D}$ a countable set of functions which is dense in $L^2(\Omega \times Y^1)$ and consists of finite sums of the form

$$\Phi(\boldsymbol{x}, y^1) = \sum \phi_j(\boldsymbol{x}) h_j(y^1), \tag{8}$$

where $\phi_j \in C_0^\infty(\Omega),\ h_j \in C_{\mathrm{per}}^\infty(Y^1)$.

For any test function of the form (8), using Young's inequality, we have

$$2\int_\Omega u_\epsilon(\boldsymbol{x}) \Phi\left(\boldsymbol{x}, \frac{x^1}{\epsilon}\right) \mathrm{d}x \leq \int_\Omega |u_\epsilon(\boldsymbol{x})|^2 \,\mathrm{d}x + \int_\Omega \left|\Phi\left(\boldsymbol{x}, \frac{x^1}{\epsilon}\right)\right|^2 \mathrm{d}x.$$

Letting $\epsilon \to 0$, by definition of weak two-scale convergence and the mean value property, we get

$$\liminf_{\epsilon\to 0} \int_\Omega |u_\epsilon|^2 \mathrm{d}x \geq 2 \int_{\Omega\times Y^1} u(\boldsymbol{x}, y^1)\Phi(\boldsymbol{x}, y^1) \mathrm{d}x\mathrm{d}y^1 - \int_{\Omega\times Y^1} |\Phi(\boldsymbol{x}, y^1)|^2 \,\mathrm{d}x\,\mathrm{d}y^1.$$

Now, choosing a sequence $\Phi(\boldsymbol{x}, y^1) = \Phi_k(\boldsymbol{x}, y^1)$ such that $\Phi_k \to u(\boldsymbol{x}, y^1)$ in $L^2(\Omega \times Y^1)$ as $k \to \infty$, we obtain (7). □

Recall that a function $\Phi(\boldsymbol{x}, y^1)$ on $\Omega \times Y^1$ is said to be a *Carathéodory function* if it is continuous in $\boldsymbol{x} \in \Omega$ for almost all $y^1 \in Y^1$ and measurable in $y^1$ for any $\boldsymbol{x} \in \Omega$.

Now, we formulate an important result about the extension of the class of admissible functions in the original Definition 4.1. More details and proofs can be found in [11, 12, 13].

**Lemma 4.5.** *Let $u_\epsilon \rightharpoonup u(\boldsymbol{x}, y^1)$. If $\Phi(\boldsymbol{x}, y^1)$ is a Carathéodory function and $|\Phi(\boldsymbol{x}, y^1)| \leq \Phi_0(y^1), \Phi_0 \in L^2(Y^1)$, then*

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon(\boldsymbol{x}) \Phi\left(\boldsymbol{x}, \frac{x^1}{\epsilon}\right) \mathrm{d}x = \int_{\Omega\times Y^1} u(\boldsymbol{x}, y^1)\Phi(\boldsymbol{x}, y^1)\,\mathrm{d}x\,\mathrm{d}y^1. \tag{9}$$

In particular, one can choose $\Phi(\boldsymbol{x}, y^1) = \phi(\boldsymbol{x}) h(y^1), \phi \in C_0^\infty(\Omega), h \in L^2(Y^1)$.

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