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Toward Improved Models for Decision Making in Economics

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Abstract

We elaborate upon four major revelations in statistical science toward improving traditional models for decision-making in social science, in particular in economics. These revelations concern "how to repair statistical inference procedures based on the notion of p-values?","how to correctly obtain predictive modeling?", "how to improve expected utility theory in economics?", and "how to faithfully model economic dynamics?".

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1 INTRODUCTION

If we ask "what is new in statistical science at the dawn of the 21st century?", we may not get a clear answer, instead, we get either "big data", or "data science"! and this sounds like "until now, everything is fine (as far as statistical science is concerned), so that let's face the challenging data-driven problems (which, obviously, will be important for economic applications)".

Why new things are important in any kinds of sciences, physical or social? Well, remember the dawn of the last century: from Newton mechanics to Einstein's relativity and quantum mechanics. If we just view data science as the new things to focus on, we might ignore the crucial fact that, as a common definition, data science is a combination of mathematics, statistics and computer science, so that the statistical component in it is "fine, until now". Are statisticians arming with correct tools and correct practices to participate in a multi-effort project such as data science?

Essentially, this paper is about recent revelations concerning statistical science, namely "how to repair statistical inference procedures based on the notion of p-values?", "how to correctly obtain predictive modeling?", "how to improve expected utility theory in economics?", and "how to faithfully model economic dynamics?", with the hope to make econometricians (and statisticians) aware of these important issues, and to invite them to take a serious look at current research afforts.

2 REPAIRING STATISTICAL INFERENCE

It should be well known by now that the actual crisis in science was caused mainly by the traditional use of Fisher's notion of p-value in hypothesis testing (in both Fisher and Neyman-Pearson settings), e.g., [10], [17], [29], [41], [42], [49], [52]. Unfortunately, while other "inferential procedures" in statistical science, such as point estimation and confidence or credible regions, are "logically" justified for practical uses, the frequentist approach to hypothesis testing based upon p-values has finally revealed its serious flaw, and has to be dealt with seriously.

While hypothesis testing could be placed within the more realistic setting of model selection, it is at the heart of statistical inference for "discovery of new knowledge'. While hypothesis testing can be carry out, at least logically (say, in common sense reasoning), by another approach, namely Bayesian approach (which is in fact prior to the frequentist approach, and was "overturned" by the frequentist approach only because of the addition of subjective prior information into the process, an addition that frequentists considered as "unscientific"), the frequentist approach still dominates (huge majority) the teaching and practice of statistical inference. Remember, an inference is not based on some mathematical theorems (as opposed to consistency of point estimators), but simply on reasoning (or logic). In other words, an inference procedure is a model for decision-making. As such, its validity should be judged on logical grounds.

While it is clear that inference based on p-values is illogical, it is still difficult to convince the statistical community as a whole to abandon it, so that we can properly participate in social science problems where statistics has an essential role to play, such as data science. This section is designed specifically to address this embarrassing situation, once for all.

Staying with p-values or resisting against its flaw, is it by "tradition"? Maybe not. If we talk about tradition, it should be Bayesian testing, and not frequentist testing! It is not by tradition, it simply steps on so many toes. Debrouwer [17] put it simply "In 1974, Richard Feynman expressed his worry about scientists who follow all the forms of science, but not its spirit, and compared this kind of science to a cult [25]. What he described is painfully close to the ritualized statistics of null hypothesis significance testing". For information about the "Cargo Cult Statistics", see [21], [25], and the recent article "Cargo-cult statistics and the scientific crisis" by P.B. Stark and A. Saltelli (July 2018), posted in Science.

Despite concrete evidence pointing to the wrong doing of p-values for quite some times, including [53], and especially ASA [52] with the clear message "By itself, a p-value does not provide a good measure of evidence regarding a model or hypothesis", there are statisticians trying to "defense" p-values, e.g. [35], and those who tried to "save p-values" by pushing the so-called significance level 0.05 to 0.005 without recognizing that, as shown below, the flaw of p-values is not because of its thresholds. Continuing using p-values in testing has created a mess in important situations such as simultaneous testings, e.g. [22].

Let's be clear, the notion of p-value, as a measure of surprise, is useful (for decision-making), but not enough (by itself) to help us to reach a definite decision. What else do we need to add to the p-value evidence to reasonably reach a decision? Perhaps Minimum Bayes factors? or more honest and transparent information in the works surrounding a testing problem?

But why the "logic" underlying the use of p-values in decision-making is not logical?. Nowhere in any textbooks that such a reasonable question was asked, upfront, except one textbook [26] where the authors said (p. 480) "At this point, the logic of the z-test can be seen more clearly. It is an argument by contradiction, designed to show that the null hypothesis will lead to an absurd conclusion and must therefore be rejected". Then followed in the last paragraph of the last chapter (A closer look at tests of significance) of their textbook by (p. 562-563) "Nowadays, tests of significance are extremely popular. One reason is that the tests are part of an impressive and well-developed mathematical theory. Another reason is that many investigators just cannot be bothered to set up chance models. The language of testing makes it easy to bypass the model, and talk about "statistical significance" results. This sounds so impressive, and there is so much mathematical machinery clanking around in the background, that tests seem truly scientific-even when they are *complete nonsense*- St. Exupery understood this kind of problem very well:

"When a mystery is too overpowering,
one dare not disobey"
(The Little Prince [3])

See more writings on statistics from D. Freedman, a probabilist turned statistician, in [27], [28].

An argument by contradiction is what statisticians labeled as "inverse probability" argument. Let's be clear. In mathematics, where the underlying logic is a two-valued (true (1) or false (0)), the proof by contradiction (called "modus tollens" in logic) is this. In Boolean logic, propositions (true ot false) are represented as subsets of a set. The (material) implication $(A \Longrightarrow B) = B^c \cup A$ is equivalent to $B^c \Longrightarrow A^c$, since

$$\begin{aligned}(B^c \Longrightarrow A^c) &= (A^c)^c \cup B^c \\ &= A \cup B^c = B^c \cup A = (A \Longrightarrow B)\end{aligned}$$

Thus, given $A \Longrightarrow B$ (an implication which is true, i.e. having truth value 1, such as in "if a function $f(.)$ is differentiable, then $f(.)$ is continuous"), we can "prove" that $B^c \Longrightarrow A^c$ with truth value 1 (here, "if $f(.)$ is not continuous, then $f(.)$ is not differentiable"). The modus tollens is valid only in two-valued logic.

In [35], the author specifically spelled out how the above modus tollens was modified to provide the "logic" for p-values, in "defense of p-values", as follows: "Statisticians are willing to pay some chance of error to extract knowledge using induction as follows: If, given $(A \Longrightarrow B)$, then the existence of a small ϵ such that $P(B) < \epsilon$ tells us that A is probably not true". This is translated into testing setting as : take $A = H_o$, take $\epsilon = \alpha$. Assume that H_o is true. If we observe X with $P(X|H_o) < \alpha$, then we "can" infer that H_o can be rejected as improbable. In other words, if H_o is true, then $P(X|H_o)$ should be "not small"; so that, by "inverse probability reasoning", if $P(X|H_o) < \alpha$, i.e., $P(X|H_o)$ is small (using a defuzzification of the fuzzy set "small" on the unit interval), they "infer" the negation of H_o. Is this a valid logic? Well, consider "if birds then fly", which is a "rule" with exceptions in artificial intelligence, a rule with truth value not necessarily equal to 1. In fact, the rule is "by default, birds fly". Now penguins do not fly, then penguins are not birds?! Here, we cannot pretend that the rule has no exceptions and apply classical logic. If you roll a die twice and observe $(6, 6)$, don't you dare to conclude that the die is not fair? As we have said earlier, a small p-value $P(X|H_o)$ only gives us a measure of surprise (if the die is fair), but by itself (as in ASA's statements [52]), it cannot help us to reach a decision. We should not view a "small" p-value as an "evidence without doubt" against H_o (leading to the rejection of H_o). In summary, the notion of p-value, as a model for decision-making, should be abandoned, or at least repaired. We need a logical inference procedure.

There are several proposed alternatives to p-values, e.g., [2], [6], [11], [16], [39], [36], [38]. As far as participating in data science is concerned, the Bayesian testing (and model selection) in Bayesian statistics seems attractive, especially when using in machine learning (see next section). Roughly speaking, while $P(X|H_o)$ cannot be used to make a decision on H_o, $P(H_o|X)$ can. Specifically, the use of the Bayes factor, as another model for decision-making, is logical, at least in common sense reasoning in artificial intelligence. Thus, repairing the frequentist testing "tradition" by teaching widely Bayesian statistics seems to be a positive step in education. In any case, the knowledge of bayesian statistics is useful in cooperative works with computer scientists in data science.

3 IMPROVING PREDICTIVE MODELING

Continuing with a specific, and important task that econometricians will definitely involve, namely cooperating in data science research with economic applications in mind, let's discuss (or remind) another "issue" in traditional statistical practices which needs to be repaired and enlarged as well. And that is the confusion that explanation modeling is the same as prediction modeling (as exemplified in regression models). A clarification of the difference between these two modeling processes, as well as an invitation to machine learning in computer science (which definitely plays an important role in data science) are provided by [9] and [47] at the turn of the century.

By tradition (again!), textbooks in statistics present regression analysis as the topic for prediction, with statements such as "Regression analysis is a statistical methodology that utilizes the relation between variables so that a response can be *predicted* from the others". In other words, what students have learned is this. Given data on variables, it suffices to model the data-generating mechanism, say by a goodness-of -fit by linear regression (by least squares), and then use it to make predictions. This teaching subsumes that the explanation modeling (the fitted linear model) is all we need, either for explaining the causal mechanisms that give rise to the data, or for predicting future outcomes. We kill two birds with one stone! And this "tradition" ignores warnings such as "regression models should not be used for extrapolation (extension a model beyond the range of the data used to fit it), because it will lead to error", but does preserve "correlation is not causation"!

The message in [9], [47] is this. To explain and to predict are two different things, and hence they need two different modeling processes. An important example of a predictive modeling is neural networks in machine learning (which can be viewed as a "model-free" approach). In other words, it is about time for statisticians to recognize that they need to enlarge their traditional "toolkit" to investigate real world problems, such as including Bayesian statistics and machine learning algorithms in their research, rather than just staying with their gold standard of classical inference (which has been intimately linked to testing and drawing conclusions from data guided by p-values). Above all, inference and prediction should be well understood. Remember, statistics is not just a set of formulas and steps to follow!

Our focus in this section is pointing out the following. Although things might seem somewhat obvious, it takes some extraordinary "actions" to change people mind for the better: a ban of NHST with p-values in [49] triggered ASA to provide guidelines for handling p-values in "traditional" statistical inference, and a thought-provoking paper like [9] to get our attention on the necessity to distinguish between explanatory models and the predictive models. Both activities are healthy in scientific spirit and should improve the effectiveness of statistical science in any integrated efforts in social sciences, such as economics.

4 IMPROVING EXPECTED UTILITY FOR BEHAVIORAL ECONOMICS

David Kreps concluded his book "Notes on the Theory of Choice" [37] (p. 198), in 1988, by the following paragraph, referring to violations of standard models for decision-making under uncertainty (von Neumann [50] and Savage [45]) from data as experimental evidence (noting that, to validate a decision model, psychologists "test that hypothesis", not by using p-values, but by observations, just like "tests" in physical science):

"These data provide a continuing challenge to the theorist, a challenge to develop and adapt the standard models so that they are more descriptive of what we see. *It will be interesting to see what will be in a course on choice theory in ten or twenty years time".*

Well, that was in 1988, what do we expect to write now a new textbook on choice theory for business and economic students, 30 years later?

This section aims at providing food for thought toward an answer to this question.

After the success of (natural) physical sciences (Newton, Einstein, quantum mechanics), it was about time to move on to social sciences, especially economics. In fact, an investigation into "the laws of thought" has been started with Boole [8] much earlier. After providing the mathematics for quantum mechanics [50], von Neumann moved on to establish the mathematics of quantitative economics [51] in which the central ingredient is his notion of expected utility, as a model of human decision-making (under uncertainty), a model that later statisticians called statistical decision theory, e.g. [13]. Von Neumann's model is widely used, of course, in game theory, in all problems involving risk analysis, e.g., [5].

Two things need to be emphasized in von Neumann's expected utility model. First, the model is for "rational" agents. It is not wrong, it is an approximation (just like Newtonian mechanics is an approximation of general relativity). It is an approximation to something else we are going to reveal! With respect to rationality, here is what Stephen Hawking observed [33], (p. 47):

"Economics is also an effective theory, based on the notion of free will plus the assumption that people evaluate their possible alternative courses of action and choose the best. That effective theory is only moderately successful in predicting behavior because, as we all know, decisions are often not rational or are based on a defective analysis of the consequences of the choice. *That is why the world is in such a mess.*"

Second, the mathematical concept of expected utilily is based upon Kolmogorov's formalism of general probability theory. It is interesting to note that the mathematical foundations of quantum mechanics [50] that he helped to develop provide the setting for quantum probability theory, but von Neumann did not consider quantum probability in his expected utility theory (say, using non-commutative integral theory). Perhaps, there was no compelling reason at the time to consider quantum probability even it is simply a generalization of Kolmogorov (commutative) probability theory. A little later, when Richard Feynman came to the Berkeley Symposium on Probability and Mathematical Statistics in 1951 ([24]) to give a talk to let probabilists and theoretical statisticians know that, while the concept of chance is the same, the calculus of probabilities in quantum mechanics is different than that of standard probability calculus, there was no reaction in the probability community. Again, this could be due to the fact that there was no "compelling" reason why such "strange" probability calculus could have a place in, say, "everyday applied statistics". Well, it has to wait untill 2013 for A. Gelman and M. Bethancourt [30] to answer it positively in "Does quantum uncertainty have a place in everyday applied statistics?" by "the generalized probability theory suggested by quantum physics might very well be relevant in the social science". And, recently, quantum probability is classified as "physical chance", in "Ten Great Ideas about chance", [20]. And so, as we will see, "Nothing is so powerful as an idea whose time has come" (Victor Hugo).

Note also that, instead, at that time, only few probabilists were interested in quantum probability, e.g., P.A. Meyer [40] and K. R. Partharsarthy [43], purely from a mathematical standpoint. In fact, Meyer's seminar is very specific, namely "Quantum Probability for Probabilists".

Expected utility is a "model" for decision-making in the sense that it could model how people make decisions in the face of uncerainty, especially in economic environment. The validity of a such model should be tested by psychologists since psychology is the science devoted to human behavior and is based upon the thesis that humans have free will so that they do not obey physical laws. It turns out that von Neumann's expected utulity model was violated in experiments, e.g., [1], and [23], leading to intensive research ever since in order to extend it to more realistic models, especially those involving non-additive models such as Choquet integral expected utility. All such efforts aim at relaxing the additivity of (Kolmogorov, or standard) probability measures, i.e., replacing probability measures by non-additive set functions (viewed as cognitive uncertainty measures and calculi), while staying in the classical Boolean setting. The 2017 Nobel Memorial prize in economics awarded to Richard H. Thaler is a testimony of the emergence of behavioral economics. As we will see, the consideration of quanum probability will definitely advance behavioral economics.

Recent experimental works, e.g. [12], [31], [34], exhibit even more. Not only that cognitive uncertainty measures are non-additive, they are non-commutative, non increasing. As such, the Boolean setting for defining uncertainty measures is no longer suitable. Nothing is new under the sun? Does this sound familiar? Remember Richard Feynman in 1951? (recently, we have remembered his "Cargo cult" of 1974!). Although, economic analyses follow classical physics so closely, it is not clear why economists should think about quantum mechanics (more in the next section), in particular, quantum probability calculus. But, we got it for free! Quantum probability, [12], [31], [40], [43], turns out to have all the "properties" that psychologists have observed in experiments. Without going into details of how to build an expected utility concept based on quantum probability (instead of Kolmogorov probability) to improve von Neumann's original decision-making model, in this paper, we think it is more appropriate to invite econometrcians to take a serious look at quantum probability first.

What is quantum probability? Well, it is simply a generalization of standard probability (a commutative one) to a non-commutative probability. Without evoking a general road map in non-commutative geometry (a la Alain Connes), we follow David Hilbert's advice "What is clear and easy to grasp attracts us, complications deter" to elaborate to statisticians on how to extend a commutative concept to a non-commutative one.

Just like extending real numbers to complex numbers, or ordinary sets to fuzzy sets, the procedure is simple. If we cannot extend a concept A directly to a desired concept B, we seek an equivalent concept C to A, which can be extended to B.

Let's first consider the simplest case of Kolmogorov probability, namely the finite sample space, representing a random experiment with a finite number of possible outcomes, e.g., a roll of a pair of dice. A finite probability space is a triple $(\Omega, \mathcal{A}, P)$ where $\Omega = \{1, 2, ..., n\}$, say, i.e., a finite set with cardinality n, $\mathcal{A}$ is the power set of Ω (events), and $P : \mathcal{A} \to [0, 1]$ is a probability measure ($P(\Omega) = 1$, and $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \varnothing$). Note that since Ω is finite, the set-function P is determined by the density $\rho : \Omega \to [0, 1]$, $\rho(j) = P(\{j\})$, with $\sum_{j=1}^{n} \rho(j) = 1$. A real-valued random variable is $X : \Omega \to \mathbb{R}$. In this finite case, of course $X^{-1}(\mathcal{B}(\mathbb{R})) \subseteq \mathcal{A}$. The domain of P is the $\sigma-$field $\mathcal{A}$ of subsets of Ω (events) which is Boolean (commutative: $A \cap B = B \cap A$), i.e., *events are commutative*, with respect to intersection of sets. We wish to generalize this setting to a non commutative one, where "extended" events could be, in general, non commutative, with respect to an "extension" of $\cap$.

For this, we need some appropriate equivalent representation for all elements in this finite probability setting. Now since $\Omega = (1, 2, ..., n\}$, each function $X : \Omega \to \mathbb{R}$ is identified as a point in the (finitely dimensional Hilbert) space $\mathbb{R}^n$, namely $(X(1), X(2), ..., X(n))^t$, which, in turn, is equivalent to a $n \times n$ diagonal matrix with diagonal terms $X(1), X(2), ..., X(n)$. and zero outside (a special symmetric matrix), i.e.,

$$X \Longleftrightarrow [X] = \begin{bmatrix} X(1) & & & & 0 \\ & X(2) & & & \\ & & 0 & . & 0 \\ & & & 0 & . \\ 0 & & & & X(n) \end{bmatrix}$$

The set of such matrices is denoted as $\mathcal{D}_o$ which is a commutative (with respect to matrix multiplication) subalgebra of the algebra of all $n \times n$ matrices with real entries. As matrices act as (bounded, linear) operators from $\mathbb{R}^n \to \mathbb{R}^n$, we have transformed (equivalently) random variables into operators on a Hilbert space.

In particular, for each event $A \subseteq \Omega$, its indicator function $1_A : \Omega \to \{0,1\}$ is identified as an element of $\mathcal{D}_o$ with diagonal terms $1_A(j) \in \{0,1\}$.As such, each event A is identified as a (orthogonal) projection on $\mathbb{R}^n$, i.e., an operator T such that $T = T^2 = T^*$ (its transpose/ adjoint). Finally, the density $\rho : \Omega \to [0,1]$ is identified with the element $[\rho]$ of $\mathcal{D}_o$ with nonnegative diagonal terms, and with trace $tr([\rho]) = 1$. An element of $\mathcal{D}_o$ with nonnegative diagonal terms is a positive operator, i.e., an operator T such that $< Tx, x > \geq 0$, for any $x \in \mathbb{R}^n$ (where $< ., . >$ denotes the scalar product of $\mathbb{R}^n$). Such an operator is necessarily symmetric (self adjoint). Thus, a probability density is a positive operator with unit trace. Thus, we have transformed the standard (Kolmogorov) probability space $(\Omega, \mathcal{A}, P)$, with $\#(\Omega) = n$, into the triple $(\mathbb{R}^n, \mathcal{P}_o, \rho)$, where $\mathcal{P}_o$ denotes the subset of projections represented by elements of $\mathcal{D}_o$ (i.e., with $0-1$ diagonal terms) which represent "ordinary" events; and ρ (or $[\rho]$), an element of $\mathcal{D}_o$, is a positive operator with unit trace.

Now, keeping $\mathbb{R}^n$ as a finitely dimensional Hilbert space, we will proceed to extend $(\mathbb{R}^n, \mathcal{P}_o, \rho)$ to a non commutative "probabilty space". It suffices to extend $\mathcal{D}_0$, a special set of symmetric matrices, to the total set of all $n \times n$ symmetric matrices, denoted as $\mathcal{S}(\mathbb{R}^n)$, so that a random variable becomes an "observable", i.e., a self-adjoint operator on $\mathbb{R}^n$; an "quantum event" is simply an arbitrary projection on $\mathbb{R}^n$, i.e., an element of $\mathcal{P}$ (the set of all projections); and the probability density ρ becomes an arbitrary positive operator with unit trace. The triple $(\mathbb{R}^n, \mathcal{P}, \rho)$ is called a (finitely dimensional) *quantum probability space*. We recognize that quantum probability is based upon a new language, not real analysis, but functional analysis (i.e., not on the geometry of $\mathbb{R}^n$, but on its non commutative geometry, namely linear operators on it).

Clearly, in view of the non commutativity of matrix multiplication, quantum events (i.e., projection operators) are non commutative, in general.

Let's pursue a little further with this finite setting. When a random variable $X : \Omega \to \mathbb{R}$ is represented by the matrix $[X]$, its possible values are on the diagonal of $[X]$, i.e., the range of X is $\sigma([X])$, the spectrum of the matrix (operator) $[X]$. For $A \subseteq \Omega$, $\Pr(A)$ is taken to be $P([1_A]) = \sum_{j \in A} \rho(j) = tr([\rho][1_A])$. More generally, $EX =$

$tr([\rho][X])$, exhibiting the important fact that the concept of "trace" (of matrix/operator) replaces integration, a fact which is essential when considering an infinitely dimensional (complex, separable) Hilbert space, such as $L^2(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3), dx)$ of squared integrable, complex-valued functions.

The spectral measure of a random variable X, represented by $[X]$, is the projection-valued "measure" $\zeta_{[X]} : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}^n) : \zeta_{[X]}(B) = \sum_{X(j)\in B} \pi_{X(j)}$, where $\pi_{X(j)}$ is the (orthogonal) projection on the space spanned by $X(j)$. From it, the "quantum" probability of the event $(X \in B)$, for $B \in \mathcal{B}(\mathbb{R})$ is taken to be $P(X \in B) = \sum_{X(j)\in B} \rho(j) = tr([\rho]\zeta_{[X]}(B))$.

The extension of the above to arbitrary $(\Omega, \mathcal{A}, P)$ essentially involves the replacement of $\mathbb{R}^n$ by an infinitely dimensional, complex and separable Hilbert space H. For details, see texts [40], [43].

We have stated several times that quantum probability is non commutative and non additive. We will make these properties more explicit now.

Recall that a quantum probability space is a triple $(H, \mathcal{P}(H), \rho)$, where $\mathcal{P}(H)$ plays the role of quantum events, and for $p \in \mathcal{P}(H)$, its probability is given by $tr(\rho p)$. Recall that observables are self adjoint operators on H, i.e., elements of $\mathcal{S}(H)$.

The probability measure $\mu_\rho(.) = tr(\rho.)$ on $\mathcal{P}(H)$ is clearly non commutative in general, since, for $p, q \in \mathcal{P}(H)$, they might not commute, i.e., $pq \neq qp$, so that $tr(\rho pq) \neq tr(\rho qp)$. Of course, that extends to non commuting observables as well.

At the experiment level, the surprising non additivity of probability is explained by the interpretation of the Schrodinger wave function $\psi(x,t)$ as a probability amplitude, i.e., the probability of finding an electron in a neighborhood dx of $\mathbb{R}^3$ (at time t) is $|\psi(x,t)|^2 dx$. The well-known two-slit experiment reveals that, for two distinct holes A and B, the probability of finding electrons when only A is open is $P_A = |\psi_A(x,t)|^2 dx$, and for B only open, $P_B = |\psi_B(x,t)|^2 dx$. When both holes are open, waves interference leads to $\psi_{A\cup B}(x,t) = \psi_A(x,t) + \psi_B(x,t)$, so that $P_{A\cup B} = |\psi_{A\cup B}(x,t)|^2 = |\psi_A(x,t) + \psi_B(x,t)|^2 \neq |\psi_A(x,t)|^2 + |\psi_B(x,t)|^2$.

It can be also seen from the probability measure $\mu_\rho(.) = tr(\rho.)$ on $\mathcal{P}(H)$. First, $\mathcal{P}(H)$ is not a Boolean algebra. It is a non distributive lattice, instead. Indeed, in view of the bijection between projections and closed subspaces of H, we have, for $p, q \in \mathcal{P}(H)$, $p \wedge q$ is taken to be the projection corresponding to the closed subspace $\mathcal{R}(p) \cap \mathcal{R}(q)$, where $\mathcal{R}(p)$ denotes the range of p; $p \vee q$ is the projection corresponding to the smallest closed subspace containing $\mathcal{R}(p) \cup \mathcal{R}(q)$. You should check $p \wedge (q \vee r) \neq (p \wedge p) \vee (p \wedge r)$, unless they commute.

On $(H, \mathcal{P}(H), \rho)$, the probability of the event $p \in \mathcal{P}(H)$ is $\mu_\rho(p) = tr(\rho p)$, and if $A \in \mathcal{S}(H)$, $\Pr(A \in B) = \mu_\rho(\zeta_A(B)) = tr(\rho\zeta_A(B))$, for $B \in \mathcal{B}(\mathbb{R})$, where ζ_A is the spectral measure of A (a projection-valued measure on $\mathcal{B}(\mathbb{R})$). With its spectral decompostion $A = \sum_{\lambda\in\sigma(A)} \lambda P_\lambda$, the distribution of A on $\sigma(A)$ is $\Pr(A = \lambda) = \mu_\rho(\rho P_\lambda)$, noting that A represents a physical quantity.

Recall that on a Kolmogorov probability space $(\Omega, \mathcal{A}, P)$, the probability is axiomatized as satisfying the additivity: for any $A, B \in \mathcal{A}$;

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Now, on $(H, \mathcal{P}(H), \rho)$, where the "quantum probability" Q (under ρ), defined as, for the "quantum event" $p \in \mathcal{P}(H)$, $Q(p) = tr(\rho p)$, does **not** , in general, satisfy the analogue, for arbitrary $p, q \in \mathcal{P}(H)$,

$$Q(p \vee q) = Q(p) + Q(q) - Q(p \wedge q)$$

i.e., $Q(.)$ is not additive. This can be seen as follows. For operators $f, g \in \mathcal{S}(H)$, their commutator is defined as

$$[f, g] = fg - gf$$

so that $[f, g] \neq 0$ if f, g do not commute (i.e., $fg \neq gf$), and zero if they commute. Then, you can check that

$$[p, q] = (p - q)(p \vee q - p - q + p \wedge q)$$

exhibiting the equivalence

$$[p, q] \neq 0 \Longleftrightarrow p \vee q - p - q + p \wedge q \neq 0$$

i.e., non commutativity is equivalent to "non additivity" (of operators).

Now, as $Q(p) = tr(\rho p)$, and by additivity of the trace operator, we see that

$$p \vee q - p - q + p \wedge q = 0$$
$$\Longrightarrow tr(\rho(p \vee q - p - q + p \wedge q)) =$$
$$Q(p \vee q) - Q(p) - Q(q) + Q(p \wedge q) = 0$$

which is the analogue of additivity for the quantum probability Q, for example for p, q which commute.

The non additivity of quantum probability arises since , in general, $p, q \in \mathcal{P}(H)$ do not commute, i.e., $[p, q] \neq 0$. In other words, the non additivity of quantum probability is a consequence of the non commutativity of observables (as self adjoint operators on a Hilbert space).

5 IMPROVING FINANCIAL DYNAMICS MODELING

As main approaches to financial market dynamics are criticized in the literature, mainly because they ignore human factors (in the modeling process), we report here a current effort (e.g., [14], [15], [32], [48]) of placing financial dynamics within an appropriate quantum mechanics formalism, namely, Bohmian mechanics, [7], to improve market dynamics modeling. Note that, although the focus is on the modeling of the dynamics of financial data, the analysis can be extended to any types of data in which the sources of their dynamics involve humans.

What we are aiming at is this. In taking into account of all necessary sources causing the fluctuations of our observed data, we should get a more faithful modeling of their "real" dynamics, leading obviously to better predictions. Again, just in the case of "quantum decision-making", we will be in the context of quantum mechanics. Upfront, this last point has been noticed previously, e.g., [4], [46].

As Stephen Hawking reminded us generously several times that economic predictions (which are the main goal of econometrics) are only moderately succssful, when we use actual statistical methods to model the data dynamics (i.e., to suggest statistical models for time series of, say, financial data), e.g., [39], perhaps, we "forgot" to include some other sources causing their dynamics in our modeling processes? An amazing analogy with physics could be this. In modeling the dynamics of falling objects to earth, we should not "forget" that air exists! See also [18], [19], [44]. We elaborate next on how to apply quantum mechanics to bulding financial models.

When citing economics as an effective theory, Hawking [33] gave an example similar to quantum mechanics in view of the free will of humans, as a counterpart of the intrinsic randomness of particles. Now, the "official" view of quantum mechanics is that dynamics of particles is provided by a "quantum law" (via the Schrodinger's wave equation), thus it is expected that some "counterpart" of the quantum law (of motion) could be found to describe economic dynamics, based upon the fact that under the same type of uncertainty (quantified by noncommutative probability) the behavior of subatomic particles is similar to that of firms and consumers.

But upfront, what we have in mind is this. Taking finance as the setting, we seek to model the dynamics of prices in a more comprehensive way than traditionally done. Specifically, besides "classical" fluctuations, the price dynamics is also "caused" by mental factors of economic agents in the market (by their free will which can be described as "quantum stochastic"). As such, we seek a dynamical model having these both uncertainty components. It will be about the dynamics of prices, so that we are going to "view" a price as a "particle", so that price dynamics will be studied as quantum mechanics (the price at a time is its position, and the change in price is its speed).

So let's see what quantum mechanics can offer? Without going into to details of quantum mechanics, it suffices to note the following. In the "conventional" view, unlike macroobjects (in Newtonian mechanics), particles in motion do not have trajectories (in their phase space), or put it more specifically, their motion cannot be described (mathematically) by trajectories (because of the Heisenberg's uncertainty principle). The dynamics of a particle with mass m is "described" by a wave function $\psi(x,t)$, where $x \in \mathbb{R}^3$ is the particle position at time t, which is the solution of the Schrodinger's equation (counterpart of Newton's law of motion of macroobjects):

$$ih\frac{\partial \psi(x,t)}{\partial t} = -\frac{h^2}{2m}\Delta_x \psi(x,t) + V(x)\psi(x,t)$$

and where $f_t(x) = |\psi(x,t)|^2$ is the probability density function of the particle position X at time t, i.e., $P_t(X \in A) = \int_A |\psi(x,t)|^2 dx$.

But, our price variable does have trajectories! Its is "interesting" to note that, we used to display financial prices fluctuations (data) which look like paths of a (geometric) Brownian motion. But Brownian motions, while having continuous paths, are nowhere differentiable, and as such, there are no derivatives to represent velocities (the second component of a "state" in the phase space)!

Well, we are lucky since there exists a non-conventional formulation of quantum mechanics, called Bohmian mechanics [7] in which it is possible to consider trajectories for particles! The following is sufficient for our discussions here.

Remark. Before deriving Bohmian mechanics and using it for financial applications, the following should be kept in mind. For physicists, Schrodinger's equation is everything: the state of a particle is "described" by the wave function $\psi(x,t)$ in the sense that the probability to find it in a region A, at time t, is given by $\int_A |\psi(x,t)|^2 dx$. As we will see, Bohmian mechanics is related to Schrodinger's equation, but presents a completely different interpretation of the quantum world, namely, it is possible to consider trajectories of particles, just like in classical, deterministic mechanics. This quantum formalism is not shared by the majority of physicists. Thus, using Bohmian mechanics in statistics should not mean that statisticians "endorse" Bohmian mechanics as the appropriate formulation of quantum mechanics! We use it since, by analogy, we can formulate (and derive) dynamics (trajectories) of economic variables.

The following leads to a new interpretation of Schrodinger's equation.

The wave function $\psi(x,t)$ is complex-valued, so that, in polar form, $\psi(x,t) = R(x,t)\exp\{\frac{i}{h}S(x,t)\}$, with $R(x,t)$, $S(x,t)$ being real-valued. The above Schrodinger's equation becomes

$$ih\frac{\partial}{\partial t}[R(x,t)\exp\{\frac{i}{h}S(x,t)\}] =$$
$$-\frac{h^2}{2m}\Delta_x[R(x,t)\exp\{\frac{i}{h}S(x,t)\}]$$
$$+V(x)[R(x,t)\exp\{\frac{i}{h}S(x,t)\}]$$

from it partial derivatives (with respect to time t) of $R(x,t)$, $S(x,t)$ can be derived. Not only that x will play the role of our price, but for simplicity, we take x as one dimentional variable, i.e., $x \in \mathbb{R}$ (so that the Laplacian Δ_x is simply $\frac{\partial^2}{\partial x^2}$) in the derivation below.

Differentiating

$$ih\frac{\partial}{\partial t}[R(x,t)\exp\{\frac{i}{h}S(x,t)\}] =$$
$$-\frac{h^2}{2m}\frac{\partial^2}{\partial x^2}[R(x,t)\exp\{\frac{i}{h}S(x,t)\}]$$
$$+V(x)[R(x,t)\exp\{\frac{i}{h}S(x,t)\}]$$

and identifying real and imaginary parts of both sides, we get, respectively

$$\frac{\partial S(x,t)}{\partial t} = -\frac{1}{2m}(\frac{\partial S(x,t)}{\partial x})^2$$
$$+V(x) - \frac{h^2}{2mR(x,t)}\frac{\partial^2 R(x,t)}{\partial x^2}$$

$$\frac{\partial R(x,t)}{\partial t} = -\frac{1}{2m}[R(x,t)\frac{\partial^2 S(x,t)}{\partial x^2}$$
$$+2\frac{\partial R(x,t)}{\partial x}\frac{\partial S(x,t)}{\partial x}]$$

The equation for $\frac{\partial R(x,t)}{\partial t}$ gives rise to the dynamical equation for the probability density function $f_t(x) = |\psi(x,t)|^2 = R^2(x,t)$. Indeed,

$$\frac{\partial R^2(x,t)}{\partial t} = 2R(x,t)\frac{\partial R(x,t)}{\partial t} =$$
$$2R(x,t)\{-\frac{1}{2m}[R(x,t)\frac{\partial^2 S(x,t)}{\partial x^2}$$

$$+2\frac{\partial R(x,t)}{\partial x}\frac{\partial S(x,t)}{\partial x}]\} =$$

$$-\frac{1}{m}[R^2(x,t)\frac{\partial^2 S(x,t)}{\partial x^2}$$

$$+2R(x,t)\frac{\partial R(x,t)}{\partial x}\frac{\partial S(x,t)}{\partial x}] =$$

$$-\frac{1}{m}\frac{\partial}{\partial x}[R^2(x,t)\frac{\partial S(x,t)}{\partial x}]$$

If we stare at the equation for $\frac{\partial S(x,t)}{\partial t}$ (corresponding to the real part of the wave function in Schrodinger's equation), then we see some analogy with classical mechanics in Hamiltonian formalism.

Recall that in Newtonian mechanics, the state of a moving object of mass m, at time t, is described as $(x, m\dot{x})$ (position $x(t)$, and momentum $p(t) = mv(t)$, with velocity $v(t) = \frac{dx}{dt} = \dot{x}(t)$). The Hamiltonian of the system is the sum of the kinetic energy and potential energy $V(x)$, namely $H(x,p) = \frac{1}{2m}v^2 + V(x) = \frac{mp^2}{2} + V(x)$. From it, $\frac{\partial H(x,p)}{\partial p} = mp$, or $\dot{x}(t) = \frac{1}{m}\frac{\partial H(x,p)}{\partial p}$. Thus, if we look at

$$\frac{\partial S(x,t)}{\partial t} = -\frac{1}{2m}(\frac{\partial S(x,t)}{\partial x})^2$$

$$+V(x) - \frac{h^2}{2mR(x,t)}\frac{\partial^2 R(x,t)}{\partial x^2}$$

ignoring the term $\frac{h^2}{2mR(x,t)}\frac{\partial^2 R(x,t)}{\partial x^2}$ for the moment, i.e., the Hamiltonian $\frac{1}{2m}(\frac{\partial S(x,t)}{\partial x})^2 - V(x)$, then the velocity of this system is $v(t) = \frac{dx}{dt} = \frac{1}{m}\frac{\partial S(x,t)}{\partial x}$.

Now the full equation has the term $Q(x,t) = \frac{h^2}{2mR(x,t)}\frac{\partial^2 R(x,t)}{\partial x^2}$, coming from Schrodinger's equation, and which we call it a "quantum potential", we follow Bohm to interprete it similarly., leading to the Bohm-Newton equation

$$m\frac{dv(t)}{dt} = m\frac{d^2x(t)}{dt^2}$$

$$= -(\frac{\partial V(x,t)}{\partial x} - \frac{\partial Q(x,t)}{\partial x})$$

giving rise to the concept of "trajectory" for the "particle".

Remark. As you can guess, Bohmian mechanics (also called "pilot wave theory") is "appropriate" for modeling financial dynamics. Roughly speaking, Bohmian mechanics is this. While fundamental to all is the wave function coming out from Schrodinger's equation, the wave function itself provides only a partial description of the dynamics. This description is completed by the specification of the actual positions of the particle, which evolve according to $v(t) = \frac{dx}{dt} = \frac{1}{m}\frac{\partial S(x,t)}{\partial x}$, called the "guiding equation" (expressing the velocities of the particle in terms of the wave function). In other words, the state is specified as (ψ, x). Regardless of the debate in physics about this formalism of quantum mechanics, Bohmian mechanics is useful for economics! Note right away that the quantum potential (field) $Q(x,t)$, giving rise to the "quantum force" $-\frac{\partial Q(x,t)}{\partial x}$, disturbing the "classical" dynamics, will play the role of "mental factor" (of economic agents) when we apply Bohmian formalism to economics.

With the fundamentals of Bohmian mechanics in place, you are surely interested in a road map to economic applications!

The "Bohmian program" for applications is this. With all economic quantities analogous to those in quantum mechanics, we seek to solve the

Schrodinger' s equation to obtain the (pilot) wave function $\psi(x,t)$ (representing expectation of traders in the market), where $x(t)$ is, say, the stock price at time t; from which we obtain the mental (quantum) potential $Q(x,t) = \frac{h^2}{2mR(x,t)} \frac{\partial^2 R(x,t)}{\partial x^2}$ producing the associated mental force $-\frac{\partial Q(x,t)}{\partial x}$; solve the Bohm-Newton's equation to obtain the "trajectory" for $x(t)$. Note that, the quantum randomness is encoded in the wave function via the way quantum probability is calculated, namely, $P(X(t) \in A) = \int_A |\psi(x,t)|^2 dx$. Of course, economic counterparts of quantities such as m (mass), h (the Planck constant) should be spelled out (e.g., number of shares, price scaling parameter, i.e., the unit in which we measure price change). The potential energy describes the interactions among traders (e.g., competition) together with external conditions (e.g., price of oil, weather, etc...) whereas the kinetic energy represents the efforts of economic agents to change prices. For some recent empirical works, see [48], [32].

Remark. When data (including economic data) are available, we look at them just as a sample of a dynamic process, i.e., just examining on how they fluctuated, and not paying enough attention on where they came from. In other words, when conducting empirical research, regardless whether data are "natural phenomenon" data or data having also some "cognitive" components (e.g., decisions from economic agents/ investors, traders in markets), we treat them the same way. Having looked at data this way, we proceed (by tradition) simply by proposing stochastic models to model their dynamics (for explanation and then prediction), such as the well-known Black-Scholes model in financial econometrics. Clearly the geometric Brownian motion model (describing the stochastic dynamics of asset prices) captures randomness of natural phenomena, but does not incorporate anything related to the effects of economic agents who are in fact responsible for the fluctuations of the prices under consideration. As such, does a "traditional" stochastic model in econometrics really describe the dynamics on which all conclusions will be derived?

Stephen Hawking nicely reminded us that, following natural sciences (i.e., physics), we should view economics (a social science) as an "effective theory", i.e., there is another important factor to take into account when proposing a model (not a "law" yet!) for dynamics of economic variables, and that is decisions of economic agents ("thinking individuals", from the existence of their free will). Whether or not, partially because of this that behavioral economics started getting attention of researchers. Of course, the problem arises because, so far, unlike, say, quantum mechanics, predictions in economics were not that successful (!), as Hawking nicely qualified it as "moderate". Should we ask "why?".

For example, financial econometrics is dominated by the so-called "efficient market hypothesis" under the influence of P.A. Samuelson and E.F. Fama, which is based upon the "assumption" that investors act rationally and without bias (and new information appears at random, and influences economic prices at random). As a consequence, using standard probability calculus, martingales are models for dynamics of asset prices, resulting in the conclusion that "trading on stock market is just a game of chance (luck) and not a game of skill", despite empirical evidence revealing that "stock dynamics is predictable to some degree".

It is all about prediction. But prediction is a consequence of our modeling process. Should we take a closer look at the way we used to model financial dynamics? Obviously, we adapt (follow) concepts and methods in natural sciences to social sciences, but not "completely". The delicate difference between Newtonian mechanics and quantum mechanics was ignored in econometrics modeling. Of course, we do not "equate" the intrinsic randomness of particle motion with the free will of economic agents's mind (in making decisions). But, if, unlike Newtonian mechanics, quantum mechanics is random so that, dynamics, trajectories of particles should be formulated differently, then the same spirit should be used in economic modeling.

But as Richard Feynman pointed out to us [24], when dealing with the randomness of particles, we need another probability calculus. Of course that was his only message to probabilists and statisticians, without knowing that later standard probability and statistics invade empirical research in economics. The quantum probability calculus seems strange (i.e., not applicable) to standard statistical practices, because quantum probability exhibits "nonadditivity" and "noncommutativity". Well, Hawking did tell us that we have to pay attention to psychologists because they are there precisely to help econometricians! Both nonadditivity and noncommutativity of a measure of fluctuations were discovered by psychologists, invalidating expect utility in the first place. The shift to nonadditive measures (in human decision-making affecting economic data) has been started long time ago, but it looks like a separate effort only for decision theory, with no incorporation into econometrics analysis. As pointed out in this present paper, nonadditive measures, such as Choquet capacities, are not adequate as a measure of fluctuations (of economic data) since they are still increasing set functions, and commutative. It is right here that we should follow physics "completely" by using quantum probability calculus in economic analysis. Recent literature shows promising research in this direction. Our hope, in an exposition such as this, is that those econometricians who are not yet aware of this revolutionary vision, will to start to consider it seriously.

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