PROPERTIES OF POSITIVE SOLUTIONS FOR FRACTIONAL MULTIPOINT BVPs AT RESONANCE

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Received: 27 June 2021; Accepted: 21 July 2021

ABSTRACT

The paper aims to study various sufficient conditions for the existence of positive solutions to the fractional differential equation

$$D^{\alpha}u(t) = f(t,u),\ 0 < t < 1,\ 1 < \alpha \le 2,$$

subject to the multipoint boundary condition

$$u'(0) = 0,\ \ u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i).$$

The main tools used are Krasnosels'kii fixed point theorem, Leggett-Williams fixed point theorem and the monotone iterative technique. In addition, the set of positive solutions is proved to be compact.

Keywords: Multipoint, boundary value problem, resonance, positive solution, solution set.

1. INTRODUCTION

Fractional differential equations arise in various areas of physics and applied mathematics and have been of great interest recently. It becomes a powerful tool in modeling of many physical phenomena (for instance, see [1-9] and the references therein). In this article, we consider the existence of positive solutions for fractional multipoint boundary value problem at resonance:

$$\begin{cases} D^{\alpha}u(t) = f(t,u),\ 0 < t < 1,\ 1 < \alpha \le 2, \\ u'(0) = 0,\ \ u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i), \end{cases} \tag{1.1}$$

where D^{α} is the Caputo fractional derivative, $m > 1,\ 0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1,\ \alpha_i > 0$ and $\sum_{i=1}^{m} \alpha_i = 1.$

When $\alpha = 2$ the problem (1.1) is studied with non-resonant condition $\sum_{i=1}^{m} \alpha_i < 1$ in [10], and with resonant condition $\sum_{i=1}^{m} \alpha_i = 1$ in [11, 12]. For the fractional case $\alpha \in (1,2)$, the problem at resonance is investigated in [13, 14] with the fractional derivative in Riemann–Liouville sense.

In this note, by the condition $\sum_{i=1}^{m} \alpha_i = 1$, the associated homogeneous BVP (when taking $f \equiv 0$ in the right hand side) has nontrivial solutions $u(t) = c$, for $c \in \mathbb{R}$ which the problem (1.1) is so-called resonant. To handle the existence of such resonant BVPs, authors mainly employ the Mawhin coincidence degree [15] which has been published recently [13, 14, 16-24]. Generally speaking, the problem non-resonant is easier to explore the existence of solutions, especially positive ones. For most cases of resonant ones studied previously, only the solvability is established, with no more deeper properties of solutions such as positiveness, compactness which is considered chiefly on BVPs at non-resonance [10, 25, 26].

Motivated by that and inspired the perturbation technique proposed in [27], in this paper, we will investigate the existence for positive solutions and the compactness of solution-set to problem (1.1) at resonance.

We consider the Banach spaces $C[0,1]$ and $C^2[0,1]$ equipped with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ and $\|u\|_2 = \max\{\|u\|, \|u'\|, \|u''\|\}$ respectively. Define a linear operator $L : D(L) \subset C^2[0,1] \to C[0,1]$ by setting $Lu := D^\alpha u + \beta^2 u$, where

$$D(L) = \left\{ x \in C^2[0,1] : u'(0) = 0,\ u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i), \right\}$$

and $\beta > 0$ is suitable constant such that L is invertible.

Inspired by the perturbation t Putting $g(t,u(t)) = f(t,u(t)) + \beta^2 u(t)$, we observe that u is a solution of the resonant BVP (1.1) if and only if it is a solution of the following non-resonant BVP

$$\begin{cases} D^\alpha u + \beta^2 u = g(t,u), 0 < \mathrm{t} < 1, \\ u'(0) = 0,\ u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i). \end{cases} \tag{1.2}$$

Throughout the paper, we make the following assumptions.

(H1) $\beta^2 \in \left(0, \dfrac{\alpha^2 - \alpha}{2\alpha - 1}\right]$ is a constant.

(H2) $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is a continuous function such that

$$f(t,x) \geq -\beta^2 x,\ \forall t \in [0,1],\ x \in [0,+\infty).$$

(H3) The function $f(t,x)$ is nondecreasing in x.

2. PRELIMINARIES

To begin the section we recall some definitions of the fractional calculus. See [28-30] for more details.

Definitions 2.1. Given $f:[0,1]\to\mathbb{R}$ and $\alpha>0$. Then

(i) The fractional integral of order α of the function f is given by

$$I^{\alpha}f(t):=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds \text{ for } t>0.$$

(ii) The Caputo fractional derivative of order α of f is given by

$$D^{\alpha}f(t):=I^{n-\alpha}\left(f^{(n)}\right)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds \text{ for } t>0,$$

where n is the smallest integer greater than or equal to a α.

Definitions 2.2. (i) The classical Mittag-Leffler function is defined by

$$E_{\alpha}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)} \qquad (\alpha>0),$$

where Γ stands for Gamma function.

(ii) The two-parametric Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)} \qquad (\alpha>0,\beta\in\mathbb{R}).$$

The next two lemmas are the Green function associated with BVP (1.2) and some estimates for it. The proofs can be found in [31].

Lemma 2.1. *For $h(t)\in C[0,1]$, then the problem*

$$\begin{cases} D^{\alpha}u(t)+\beta^2u(t)=h(t), \ 0<t<1, \\ u'(0)=0, \ u(1)=\sum_{i=1}^{m}\alpha_i u(\eta_i), \end{cases} \tag{2.1}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)h(s)ds, \tag{2.2}$$

where

$$G(t,s)=\begin{cases} (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\beta^2(t-s)^{\alpha}\right), & 0\le s\le t\le 1 \\ 0, & 0\le t\le s\le 1 \end{cases}$$

$$+\frac{E_\alpha\left(-\beta^2 t^\alpha\right)}{a_0}\begin{cases}(1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\beta^2(1-s)^\alpha\right)-\sum_{i=1}^{m}\alpha_i(\eta_i-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\beta^2(\eta_i-s)^\alpha\right), & 0\le s\le\eta_1,\\ (1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\beta^2(1-s)^\alpha\right)-\sum_{i=2}^{m}\alpha_i(\eta_i-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\beta^2(\eta_i-s)^\alpha\right), & \eta_1\le s\le\eta_2,\\ \vdots & \\ (1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\beta^2(1-s)^\alpha\right)-\sum_{i=k}^{m}\alpha_i(\eta_i-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\beta^2(\eta_i-s)^\alpha\right), & .\eta_{k-1}\le s\le\eta_k,\\ (1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\beta^2(1-s)^\alpha\right), & \eta_m\le s\le 1.\end{cases} \tag{2.3}$$

here $a_0=\sum_{i=1}^{m}\alpha_i E_\alpha\left(-\beta^2\eta_i^\alpha\right)-E_\alpha\left(-\beta^2\right).$

Lemma 2.2. *For given* $\beta^2\in\left(0,\frac{\alpha^2-\alpha}{2\alpha-1}\right]$, *we have that*

(i) there exist $c_1,\ c_2>0$ *such that*

$$c_1(1-s)^{\alpha-1}\le G(t,s)\le c_2(1-s)^{\alpha-1},\ \forall t,\ s\in[0,1].$$

(ii) There exist $M,\ M_0>0$ *such that*

$$0\le G(t,s)\le M,\ \ \forall t,s\in[0,1], \tag{2.4}$$

and

$$G(t,s)\ge M_0,\ \ \forall t\in[0,1],\ s\in[0,\eta_m]. \tag{2.5}$$

Now put $c=\frac{c_1}{c_2}$. Let K be a cone in $C[0,1]$ which consists of all nonnegative functions and put $P=\{u\in K: u(t)\ge c\|u\|,\ \forall t\in[0,1]\}$. Obviously, P is a cone in $C[0,1]$. For $u\in P$, denote $F(u)(t)=g(t,u(t))$ and define a map A as follows

$$Ah(t)=\int_0^1 G(t,s)h(s)ds, \text{ for } h\in C[0,1], t\in[0,1].$$

Then we have the following lemma.

Lemma 2.3. *The operator* $T=A\circ F:P\to P$ *is completely continuous.*

Proof. From (H2) we deduce that the operator $F:P\to K$ is continuous. So, the operator $T=A\circ F:P\to K$ is completely continuous. Moreover, for each $u\in P$, by Lemma 2.2-(i) we have

$$Tu(t)=\int_0^1 G(t,s)F(u)(s)ds\ge c_1\int_0^1(1-s)^{\alpha-1}F(u)(s)ds, \tag{2.6}$$

$$\|Tu\|=\max_{t\in[0,1]}\int_0^1 G(t,s)F(u)(s)ds\le c_2\int_0^1(1-s)^{\alpha-1}F(u)(s)ds \tag{2.7}$$

Combining (2.6)-(2.7) and $c=\frac{c_1}{c_2}$, we obtain

$$Tu(t) \geq c\|Tu\|. \tag{2.8}$$

Therefore, $Tu \in P$. The proof is complete.

It is noted that a nonzero fixed points of the operator T is a positive solutions of BVP (1.2).

To conclude this section, we recall some well-known fixed point theorems for main tools to establish our results in the next section.

Theorem 2.1. *(Krasnosel'skii, [32]) Let X be a Banach space, and let $P \subset X$ be a cone. Assume Ω_1, Ω_2 are two open bounded subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ and let $T : P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to P$ be a completely continuous operator such that*

(i) $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2$, or

(ii) $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

Theorem 2.6. *(Leggett-Williams, [33]) Let K be a cone in a Banach space X, $K_c = \{u \in K : \|u\| < c\}$ and γ be a nonnegative continuous concave functional on K with $\gamma(u) \leq \|u\|$ for all $u \in K_c$. Assume that $S(\gamma, b, d) = \{u \in K : b < \gamma(u), \|u\| \leq d\}$ and $T : \overline{K}_c \to \overline{K}_c$ is a completely continuous map such that there exist the constants $0 < a < b < d \leq c$ satisfying the conditions*

(i) $\{u \in S(\gamma, b, d) : \gamma(u) > b\} \neq \emptyset$ and $\gamma(Tu) > b$ for $u \in S(\gamma, b, d)$,

(ii) $\|Tu\| < a$ for $\|u\| \leq a$,

(iii) $\gamma(Tu) > b$ for $u \in S(\gamma, b, c)$ with $\|Tu\| > d$.

Then T has at least three fixed points u_1, u_2 and u_3 with

$$\|u_1\| \leq a, b < \gamma(u_2), a < \|u_3\| \text{ with } \gamma(u_3) < b.$$

3. MAIN RESULTS

We begin our results with one for the existence of one positive solution via Krasnosel'skii fixed point theorem.

Theorem 3.1. *Let (H1)-(H2) hold. Assume that there exist two constants $R_1, R_2 > 0$ such that $R_1 < cR_2$ and either*

$$\begin{cases} f(t,u) + \beta^2 u \leq \dfrac{R_1}{M}, \ \forall (t,u) \in [0,1] \times [cR_1, R_1], \\ f(t,u) + \beta^2 u \geq \dfrac{R_2}{M_0 \eta_m}, \ \forall (t,u) \in [0,1] \times [cR_2, R_2], \end{cases} \tag{3.1}$$

or

$$\begin{cases} f(t,u)+\beta^2 u \geq \dfrac{R_1}{M_0\eta_m}, \ \forall (t,u)\in[0,1]\times[cR_1,R_1], \\ f(t,u)+\beta^2 u \leq \dfrac{R_2}{M}, \ \forall (t,u)\in[0,1]\times[cR_2,R_2]. \end{cases} \tag{3.2}$$

Then BVP (1.2) has a positive solution.

Proof. Let $\Omega_1=\{u\in C[0,1]:\|u\|<R_1\}, \Omega_2=\{u\in C[0,1]:\|u\|<R_2\}$. Then Ω_1, Ω_2 are open bounded subsets of $C[0,1]$ with $0\in\Omega_1, \overline{\Omega_1}\subset\Omega_2$.

Case (3.1). For $u\in P$ with $\|u\|=R_1$, we have

$$0\leq g(t,u(t))=f(t,u(t))+\beta^2 u(t)\leq \frac{R_1}{M}=\frac{\|u\|}{M}, \text{ for } t\in[0,1]. \tag{3.3}$$

From (3.3) and (2.4) we obtain

$$\|Tu\|=\max_{t\in[0,1]}\int_0^1 G(t,s)g(s,u(s))ds\leq \frac{\|u\|}{M}\max_{t\in[0,1]}\int_0^1 G(t,s)ds\leq\frac{\|u\|}{M}.M=\|u\|.$$

Thus,

$$\|Tu\|\leq\|u\|, \forall u\in P\cap\partial\Omega_1. \tag{3.4}$$

On the other hand, for $u\in P$ with $\|u\|=R_2$, we have $u(t)\in[cR_2,R_2]$ and then

$$g(t,u(t))=f(t,u(t))+\beta^2 u(t)\geq\frac{R_2}{M_0\eta_m}, \text{ for } t\in[0,1]. \tag{3.5}$$

It follows from (3.5) and (2.5) that

$$Tu(t)=\int_0^1 G(t,s)g(s,u(s))ds\geq\frac{R_2}{M_0\eta_m}\int_0^{\eta_m}G(t,s)ds\geq R_2=\|u\|,$$

Therefore

$$\|Tu\|\geq\|u\|, \forall u\in P\cap\partial\Omega_2. \tag{3.6}$$

By (3.4), (3.6) and the first part of Theorem 2.1, *T* has a fixed point in $P\cap(\overline{\Omega_2}\setminus\Omega_1)$, that is, a positive solution to BVP (1.2).

Case (3.2). Using the similar argument and by applying the second part of Theorem 2.1, we obtain the same result as above. The theorem is proved. ■

Using the monotone iterative technique, we get the following result.

Theorem 3.2. *Let (H1) and (H3) hold. If there exist two constants* R_1, R_2 with $0<R_1<R_2$ *such that*

$$\sup_{t\in[0,1]}g(t,R_2)\leq\frac{R_2}{M}, \ \inf_{t\in[0,1]}g(t,cR_1)\geq\frac{R_1}{M_0\eta_m}. \tag{3.7}$$

Then the problem (1.2) has positive solutions u_1,u_2 *with*

$$R_1\leq\|u_1\|\leq R_2 \ \textit{and} \ \lim_{n\to+\infty}T^n u_0=u_1, \ \textit{where} \ u_0(t)=R_2, \ t\in[0,1],$$

and

$$R_1\leq\|u_2\|\leq R_2 \ \textit{and} \ \lim_{n\to+\infty}T^n u_0=u_2, \ \textit{where} \ u_0(t)=R_1, \ t\in[0,1].$$

Proof. Define $P_{[R_1,R_2]}=\{u\in P:R_1\leq\|u\|\leq R_2\}$. Suppose $u\in P_{[R_1,R_2]}$, then

$$cR_1 \le c\|u\| \le u(t) \le \|u\| \le R_2, \forall t \in [0,1].$$

From (H3) we have

$$Tu(t) = \int_0^1 G(t,s) g(s,u(s)) ds \le \int_0^1 G(t,s) g(s,R_2) ds \le \frac{R_2}{M} \int_0^1 G(t,s) ds \le R_2,$$

and

$$Tu(t) = \int_0^1 G(t,s) g(s,u(s)) ds \ge \int_0^1 G(t,s) g(s,cR_1) ds \ge \frac{R_1}{M_0 \eta_m} \int_0^1 G(t,s) ds \ge R_1.$$

Hence

$$TP_{[R_1,R_2]} \subset P_{[R_1,R_2]}. \tag{3.8}$$

Let $u_0(t) = R_2$, $t \in [0,1]$ and so $u_0 \in P_{[R_1,R_2]}$. Put

$$u_{n+1} = Tu_n = T^{n+1}u_o, \; n = 1,2,... \tag{3.9}$$

By (3.8) we have $u_n \in P_{[R_1,R_2]}$, $\forall n \ge 1$. It follows from Lemma 2.3 that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\lim_{k\to+\infty} u_{n_k} = u_1 \in P_{[R_1,R_2]}. \tag{3.10}$$

Moreover, we have

$$0 \le u_1(t) \le \|u_1\| \le R_2 = u_0(t), \forall t \in [0,1]. \tag{3.11}$$

From (H3), it is easy to see that $T : P_{[R_1,R_2]} \to P_{[R_1,R_2]}$ is nondecreasing. Therefore, (3.11) deduces that $Tu_1 \le Tu_0$ or $u_2 \le u_1$. By induction, we have

$$u_{n+1} \le u_n, \; \forall n \ge 1. \tag{3.12}$$

Combining (3.10) and (3.12) gives us

$$\lim_{k\to+\infty} u_n = u_1. \tag{3.13}$$

Letting $n \to +\infty$ in (3.9) we obtain $Tu_1 = u_1$, or u_1 is a positive solution of (1.2).

Using the same technique by setting $u_0(t) = R_1, \forall t \in [0,1]$ and considering a sequence $\{u_n\}$ defined by $u_{n+1} = Tu_n$, $n = 1,2,...$, we have $u_0 \in P_{[R_1,R_2]}$ and $u_n \in P_{[R_1,R_2]}$ and then $Tu_2 = u_2$. This completes the proof of the theorem. ■

Corollary 3.1. *Let (H1) and (H3) hold. Assume that*

$$\liminf_{u\to+\infty} \sup_{t\in[0,1]} \frac{f(t,u)}{u} \le -\beta^2 + \frac{1}{M} \quad \text{and} \quad \limsup_{u\to 0^+} \inf_{t\in[0,1]} \frac{f(t,u)}{u} \ge \frac{1}{M_0\eta_m}.$$

Then there exist two constants $R_1, R_2 > 0$ ($R_1 < R_2$) *and the problem (1.2) has positive solutions* u_1, u_2 *for which*

$$R_1 \le \|u_1\| \le R_2 \text{ and } \lim_{n\to+\infty} T^n u_0 = u_1, \text{ where } u_0(t) = R_2, \; t \in [0,1],$$

and

$$R_1 \le \|u_2\| \le R_2 \text{ and } \lim_{n\to+\infty} T^n u_0 = u_2, \text{ where } u_0(t) = R_1, \; t \in [0,1].$$

Clearly, this corollary is a direct consequence of Theorem 3.3.

Now, we will give sufficient conditions for existence of infinitely many positive solutions.

Theorem 3.3. *Let (H1)-(H2) hold and suppose that there exists a sequence* $\{R_n\}_{n=1}^{\infty} \subset \mathbb{R}$ *with* $0 < R_n < cR_{n+1}$ *and for all* $n \in \mathbb{N}$, *we have*

$$f(t,u)+\beta^2 u \le \frac{R_{2n-1}}{M}, \ \forall (t,u) \in [0,1]\times[cR_{2n-1}, R_{2n-1}], \tag{3.14}$$

$$f(t,u)+\beta^2 u \ge \frac{R_{2n}}{M_0\eta_m}, \ \forall (t,u) \in [0,1]\times[cR_{2n}, R_{2n}]. \tag{3.15}$$

Then BVP (1.2) has infinitely many positive solutions $\{u_n\}_{n\in\mathbb{N}}$ *satisfying* $R_{2n-1} \le \|u_n\| \le R_{2n}$ *for all* $n \in \mathbb{N}$.

Proof. For each n, put $\Omega_n = \{u \in C[0,1] : \|u\| < R_n\}$. Then $0 \in \Omega_n$ and $\overline{\Omega}_n \subset \Omega_{n+1}$, $\forall n \in \mathbb{N}$. For $u \in P \cap \partial\Omega_{2n-1}$ and $s \in [0,1]$, we have

$$cR_{2n-1} = c\|u\| \le u(s) \le \|u\| = R_{2n-1}. \tag{3.16}$$

Combining (3.14) and (3.16) deduces that

$$\|Tx\| = \max_{t\in[0,1]} \int_0^1 G(t,s)g(s,u(s))ds \le \frac{R_{2n-1}}{M}\int_0^1 G(t,s)ds \le R_{2n-1} = \|u\|, \ t \in [0,1],$$

that is,

$$\|Tu\| \le \|u\|, \ \forall u \in P \cap \partial\Omega_{2n-1}. \tag{3.17}$$

On the other hand, for $u \in P \cap \partial\Omega_{2n}$ and $s \in [0,1]$, we have

$$cR_{2n} = c\|u\| \le u(s) \le \|u\| = R_{2n}.$$

Hence, by (3.15),

$$Tu(t) = \int_0^1 G(t,s)g(s,u(s))ds \ge \frac{R_{2n}}{M_0\eta_m}\int_0^{\eta_m} G(t,s)ds \ge R_{2n} = \|u\|.$$

Therefore

$$\|Tu\| \ge \|u\|, \ \forall u \in P \cap \partial\Omega_{2n}. \tag{3.18}$$

By (3.17), (3.18) and the first part of Theorem 2.1, it follows that T has a fixed point u_n in $P \cup \left(\overline{\Omega}_{2n} \setminus \Omega_{2n-1}\right)$, and $R_{2n-1} \le \|u_n\| \le R_{2n}$. The proof is complete. ∎

Next we propose a different kind of sufficient conditions for the existence of three positive solutions via Leggett-Williams fixed point theorem. For this purpose we define a nonnegative continuous concave functional γ on K by setting

$$\gamma(u) = \min_{t\in[0,1]} |u(t)| \in K.$$

It is obvious to see that $\gamma(u) \le \|u\|, \ \forall u \in K.$ Then we have the following result.

Theorem 3.4. *Let (H1)-(H2) hold. Suppose that there exist positive constants* a, b *and* d *such that* $a < b < \frac{b}{c} < d$ *and* f *satisfies the following conditions.*

(G1) $f(t,u)+\beta^2 u \le \frac{d}{M}$, *for all* $(t,u) \in [0,1]\times[0,d]$,

(G2) $f(t,u)+\beta^2 u < \dfrac{a}{M}$, *for all* $(t,u)\in[0,1]\times[0,a]$,

(G3) $f(t,u)+\beta^2 u > \dfrac{b}{M_0\eta_m}$, *for all* $(t,u)\in[0,1]\times\left[0,\dfrac{b}{c}\right]$.

Then the BVP (1.2) has at least three positive solutions u_1,u_2 *and* u_3 *satisfying*

$$\|u_1\|\le a,\ b<\min_{t\in[0,1]}|u_2(t)|,\ a<\|u_3\| \text{ with } \min_{t\in[0,1]}|u_3(t)|<b.$$

Proof. First, for $u\in\overline{K}_d$ then $\|u\|\le d$. For all $t\in[0,1]$, by assumption (G1) and (2.4) we obtain

$$|Tu(t)|=\left|\int_0^1 G(t,s)g(s,u_n(s))ds\right|\le\frac{d}{M}\int_0^1|G(t,s)|ds\le\frac{d}{M}M=d.$$

Hence, $\|Tu\|\le d$ which implies $T:\overline{K}_d\to\overline{K}_d$.

Similarly, from (G2), we also have $\|Tu\|<a$ for $\|u\|\le a$.

Now we verify that $\left\{u\in S\left(\gamma,b,\dfrac{b}{c}\right):\gamma(u)>b\right\}\ne\varnothing$. Indeed, let $\varepsilon>0$ be such that $c<c+\varepsilon<1$ and put $u(t)=-\left(\dfrac{b}{c}-\dfrac{b}{c+\varepsilon}\right)t-\dfrac{b}{c+\varepsilon}$, $t\in[0,1]$. It follows that

$$\|u\|=\frac{b}{c} \text{ and } \gamma(u)=\min_{t\in[0,1]}|u(t)|=\frac{b}{c+\varepsilon}>b.$$

Thus,

$$\left\{u\in S\left(\gamma,b,\frac{b}{c}\right):\gamma(u)>b\right\}\ne\varnothing.$$

Next, we show that $\gamma(Tu)>b,\ \forall u\in S\left(\gamma,b,\dfrac{b}{c}\right)$. Indeed, for $u\in S\left(\gamma,b,\dfrac{b}{c}\right)$, we deduce from (G3) and (2.4) that

$$\gamma(Tu)=\min_{t\in[0,1]}|Tu(t)|=\min_{t\in[0,1]}\int_0^1|G(t,s)|g(s,u(s))ds>\frac{b}{M_0\eta_m}\int_0^{\eta_m}|G(t,s)|ds\ge b.$$

Finally, suppose $u\in S(\gamma,b,d)$ with $\|Tu\|>\dfrac{b}{c}$. From (2.7), we obtain

$$\gamma(Tu)=\min_{t\in[0,1]}|Tu(t)|\ge c\|Tu\|>b.$$

Hence, applying Theorem 2.2 we get T has three fixed points u_1,u_2 and u_3 with

$$\|u_1\|\le a,\ b<\min_{t\in[0,1]}|u_2(t)|,\ a<\|u_3\| \text{ with } \min_{t\in[0,1]}|u_3(t)|<b.$$

The proof is complete. ■

The last part of our results is to investigate a topological property of solution-set: compactness.

Theorem 3.5. *Let (H1)-(H2) hold and assume that there exists a constant* $\gamma\in(0,1)$ *such that*

$$f_0 \geq \frac{1}{M\eta_m} \text{ and } f^{\infty} \leq -\beta^2 + \frac{\gamma}{M}. \tag{3.19}$$

Then the set of positive solutions of the problem (1.2) is nonempty and compact.

Proof. Put $S = \{u \in P : u = Tu\}$. By Theorem 3.1, S is nonemty. Moreover, from (3.19) there has a constant $R > 0$ such that

$$f(t,u(t)) \leq \left(-\beta^2 + \frac{\gamma}{M}\right)u(t),\ \forall t \in [0,1],\ u(t) \geq R.$$

Hence

$$g(t,u(t)) = f(t,u(t)) + \beta^2 u(t) \leq \frac{\gamma}{M}u(t),\ \forall t \in [0,1],\ u(t) \geq R. \tag{3.20}$$

Setting $N = \max\{g(t,u) : (t,u) \in [0,1]\times[0,R]\}$ we get from (3.20) that

$$g(t,u(t)) \leq \frac{\gamma}{M}u(t) + N,\ \forall t \in [0,1]. \tag{3.21}$$

Let $u \in S$. By (3.21) and (2.4) we have

$$u(t) = \int_0^1 G(t,s)g(s,u(s))ds \leq M\int_0^1\left(\frac{\gamma}{M}u(s) + N\right)ds \leq \gamma\|u\| + MN, \text{ for } t \in [0,1],$$

which implies

$$\|u\| \leq \frac{MN}{1-\gamma}, \forall u \in S. \tag{3.22}$$

Using the compactness of the operator T and (3.22) we deduce that sets $T(S)$ and $S \subset T(S)$ are relatively compact. Next, we show that S is closed. To do this, we assume that $\{u_n\} \subset S$ be a sequence such that $\lim_{n\to+\infty}\|u_n - \hat{u}\| = 0$. For $t \in [0,1]$ then we have

$$\begin{aligned}\left|\hat{u}(t) - \int_0^1 G(t,s)g(s,\hat{u}(s))ds\right| &\leq |\hat{u}(t) - u_n(t)| + \left|u_n(t) - \int_0^1 G(t,s)g(s,u_n(s))ds\right| \\ &\quad + \left|\int_0^1 G(t,s)g(s,u_n(s))ds - \int_0^1 G(t,s)g(s,\hat{u}(s))ds\right| \\ &\leq |\hat{u}(t) - u_n(t)| + M\int_0^1 |g(s,u_n(s)) - g(s,\hat{u}(s))|ds.\end{aligned}$$

As g is continuous, letting $n \to +\infty$, we obtain

$$\left|\hat{u}(t) - \int_0^1 G(t,s)g(s,\hat{u}(s))ds\right| = 0,$$

which means

$$\hat{u}(t) = \int_0^1 G(t,s)g(s,\hat{u}(s))ds.$$

Thus, $\hat{u} \in S$. The proof is complete. ■

4. CONCLUSION

This report is dedicated to deal with the multiplicity of positive solutions to a Caputo-fractional multipoint BVP at resonance including the existence of one positive solutions, three positive solutions, infinitely many positive solutions, two iterative sequences conversing solutions, and the compactness of the solution-set. The method is based mainly on fixed point theorems of Krasnosel'skii and Leggett-Williams.

In a coming research, it is not trivial to consider the equation $(1.1)_1$ combining with a different kind of boundary conditions due to the difficulty from the fractional calculations.

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TÓM TẮT

CÁC TÍNH CHẤT CỦA NGHIỆM DƯƠNG CHO BÀI TOÁN BIÊN ĐA ĐIỂM BẬC PHÂN THỨ CỘNG HƯỞNG

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Bài báo nghiên cứu nhiều kiểu điều kiện đủ khác nhau cho sự tồn tại nghiệm dương của phương trình vi phân phân thứ

$$D^{\alpha}u(t) = f(t,u), \ 0 < t < 1, \ 1 < \alpha \le 2,$$

liên kết với điều kiện biên đa điểm

$$u'(0) = 0, \ \ u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i).$$

Công cụ chính được sử dụng là định lý điểm bất động Krasnosels'kii, định lý điểm bất động Leggett-Williams và kỹ thuật lặp đơn điệu. Hơn nữa, tập nghiệm dương của bài toán cũng được chứng minh là compact.

Từ khóa: Đa điểm, bài toán biên, cộng hưởng, nghiệm dương, tập nghiệm.