A UNIFIED APPROACH TO ZERO DUALITY GAP FOR CONVEX OPTIMIZATION PROBLEMS

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Article history

Received: 13/5/2021; Received in revised form: 26/7/2021; Accepted: 08/9/2021

Abstract

In this paper we establish necessary and sufficient condition for zero duality gap of the optimization problem involving the general perturbation mapping via characteringsetunder the convex setting. An application to the class of composite optimization problems will also be given to show that our general results can be applied to various classes of optimization problems.

Keywords: *Characterizing set, composite optimization problem, perturbation function, zero duality gap.*

MỘT CÁCH TIẾP CẬN THỐNG NHẤT CHO KHOẢNG CÁCH ĐỐI NGẪU BẰNG KHÔNG CHO BÀI TOÁN TỐI ƯU LỒI

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Lịch sử bài báo

Ngày nhận: 13/5/2021; Ngày nhận chỉnh sửa: 26/7/2021; Ngày duyệt đăng: 08/9/2021

Tóm tắt

Trong bài viết này, chúng tôi thiết lập điều kiện cần và đủ cho tính chất khoảng cách đối ngẫu bằng không cho bài toán tối ưu có liên quan đến hàm nhiễu tổng quát thông qua tập đặc trưng dưới giả thiết lồi. Một áp dụng cho lớp bài toán tối ưu liên quan hàm hợp cũng sẽ được trình bày để chứng tỏ rằng kết quả tổng quát của chúng tôi có thể áp dụng được cho nhiều lớp bài toán tối ưu khác nhau.

Từ khóa: *Tập đặc trưng, bài toán tối ưu hợp, hàm nhiễu, khoảng cách đối ngẫu bằng không.*

DOI: https://doi.org/10.52714/dthu.11.5.2022.975

Cite: Dang Hai Long and Tran Hong Mo. (2022). A unified approach to zero duality gap for convex optimization problems. *Dong Thap University Journal of Science, 11*(5), 9-18.

1. Introduction

It is well known that duality theory plays an important role in optimization. For a primal problem, there are different ways to define its dual problems (Feizollahi *et al*., 2017, Huang and Yang, 2003, Li, 1995, Yang and Huang, 2001). The zero duality gap is known as the state in which the optimal values of the primal problem and that of its dual problem are equal. Many attempts have been made to study the zero duality gap for various classes of optimization problems in recent decades (Feizollahi *et al*., 2017, Huang and Yang, 2003, Jeyakumar and Li, 2009a, Jeyakumar and Li, 2009b, Jeyakumar and Wolkowicz, 1990, Li, 1995, Huang and Yang, 2003, Yang and Huang, 2001, Li, 1999, Long and Zeng, 2020, Rubinov *et al.*, 2002). In this paper, we establish characterizations of zero duality gap property for the general optimization problem which can then be applied to many different specific classes optimization problems.

We are concerned with the so-called *perturbation function* $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ and the optimization problem

$$\text{(P)} \qquad \inf_{x\in X} \phi(x, 0_Y),$$

Where X, Y are locally convex Hausdorff topological vector spaces, Y_+ is non-empty convex cone in Y. We assume in this paper that $\operatorname{dom}\phi(.,0_Y) \neq \emptyset$, or in other words, the problem (P) is feasible, meaning that $v(\text{P}) < +\infty$. It is worth commenting that many classes of optimization problems can be written in the form of (P) (see Boţ, 2010). So, investigating the problem (P) gives us a unified approach to all optimization problems.

In this paper, we study characterizations of the zero duality gap property for the problem (P) via its *characterizing set* which is inspired by the concept of characterizing set introduced by Dinh *et al*. (2020) for the vector optimization with geometric and cone constrains. It is worth observing that the characterizing set is rather simpler than those sets in the form of epigraph of conjugate mapping. Therefore, the conditions imposed on the characterizing set will be easier to handle than the ones related to the epigraph of conjugate mapping proposed recently to examine the zero duality gap property (see, e.g., Jeyakumar and Li, 2009a).

The paper is organized as follows: In Section 2 we recall some notation and introduce some preliminary results which will be used in the sequel. Characterizing set and Lagrange dual problems of the problem (P) are introduced in Section 3 with related basic properties. Section 4 is devoted to establish the main results of this paper, that is, the characterization of zero duality gap for the problem (P) under the convex setting. As an illustrative example, in Section 5, we show how to apply generalized results to the classes of composite optimization problems.

2. Preliminaries

Throughout the paper, we consider X and Y the locally convex Hausdorff topological vector spaces with topological dual spaces X^* and Y^*, respectively. Y_+ is a non-empty convex cones in Y while Y_+^* aims the set of positive functionals on Y with respect to Y_+, i.e.,

$$Y_+^* := \{y^* \in Y^* : \langle y^*, k\rangle \geq 0 \quad \text{for all} \quad k \in Y_+\}.$$

Let $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. *Domain*, *epigraph*, and *hypograph* of f are defined by, respectively,

$$\operatorname{dom} f := \{x \in X : f(x) \neq +\infty\},$$
$$\operatorname{epi} f := \{(x,\alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\},$$
$$\operatorname{hyp} f := \{(x,\alpha) \in X \times \mathbb{R} : f(x) \geq \alpha\}.$$

f is said to be *proper* if $f(x) \neq -\infty$ for all $x \in X$ and $\operatorname{dom} f \neq \emptyset$. We say that f is *convex* if the following condition holds for all $x_1, x_2 \in X$ and $\alpha \in (0,1)$

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2).$$

It is easy to see that f is convex if and only if $\operatorname{epi} f$ is a convex subset of $X \times \mathbb{R}$. The conjugate function of f is defined as $f^* : X^* \to \overline{\mathbb{R}}$ such that

$$f^*(x^*) = \sup_{x\in X}[\langle x^*, x\rangle - f(x)].$$

We consider in Y the partial order induced by Y_+, $\leqq_{Y_+}$, defined as

$$y_1 \leqq_{Y_+} y_2 \quad \text{if and only if} \quad y_2 - y_1 \in Y_+.$$

We also enlarge Y by attaching a greatest element $+\infty_Y$ and a smallest element $-\infty_Y$, which do not belong to Y, and define $Y^\bullet := Y \cup \{-\infty_Y, +\infty_Y\}$. Let $H: X \to Y^\bullet$. We say that H is a Y_+-convex mapping if, for all $x_1, x_2 \in X$ and $\alpha \in (0,1)$,

$H(\alpha x_1 + (1-\alpha)x_2) \leqq_Y \alpha H(x_1) + (1-\alpha)H(x_2)$. We define the *domain* of H as $\operatorname{dom} H := \{x \in X : H(x) \neq +\infty_Y\}$ and say that H is proper if $-\infty_Y \notin H(X)$ and $\operatorname{dom} H \neq \varnothing$. When H is a proper mapping, *the image* and *the graph* of H are defined by, respectively,

$$\operatorname{im} H := \{H(x) : x \in \operatorname{dom} H\},$$
$$\operatorname{gr} H := \{(x, H(x)) : x \in \operatorname{dom} H\}.$$

We say that $g: Y \to \overline{\mathbb{R}}$ is a Y_+-*nondecreasing* function if $g(y_1) \leq g(y_2)$ whenever $y_1 \leq_Y y_2$. In the meantime, for $y^* \in Y^*$, we convention that

$$(y^* \circ H)(x) = \begin{cases} \langle y^*, H(x) \rangle & \text{if} \quad x \in \operatorname{dom} H \\ +\infty & \text{else.} \end{cases}$$

3. Characterizing set and Lagrange dual problems

3.1. Characterizing set

Corresponding to the problem (P), we consider the *characterizing set*

$$\mathfrak{C} := \bigcup_{x \in X} \operatorname{epi} \phi(x, .) \subset Y \times \mathbb{R}. \tag{3.1}$$

Proposition 3.1. *Under the current assumption* $\operatorname{dom} \phi(., 0_Y) \neq \varnothing$, *one has* $(0_Y, r) \in \mathfrak{C}$ *for some* $r \in \mathbb{R}$. *In particular,* $\mathfrak{C} \neq \varnothing$.

Proof. As $\operatorname{dom} \phi(., 0_Y) \neq \varnothing$, there exists $\bar{x} \in X$ such that $\phi(\bar{x}, 0_Y) \in \mathbb{R}$. Take $\bar{r} := \phi(\bar{x}, 0_Y) \in \mathbb{R}$, one has $(0_Y, \bar{r}) \in \operatorname{epi} \phi(\bar{x}, .) \subset \mathfrak{C}$, and we are done. □

The convexity of $\mathfrak{C}$ is shown in the following proposition.

Proposition 3.2. *If* ϕ *is convex then* $\mathfrak{C}$ *is a convex subset of* $Y \times \mathbb{R}$.

Proof. We begin by proving that $\mathfrak{C}$ is image of the set $\operatorname{epi} \phi$ by the conical projection $\pi_{Y \times \mathbb{R}} : X \times Y \times \mathbb{R} \to Y \times \mathbb{R}$, $\pi_{Y \times \mathbb{R}}(x, y, r) = (y, r)$ for all $(x, y, r) \in X \times Y \times \mathbb{R}$. Indeed, for all $(y, r) \in Y \times \mathbb{R}$,

$$\begin{aligned} (y, r) \in \mathfrak{C} &\Leftrightarrow \exists x \in X : (y, r) \in \operatorname{epi} \phi(x, .) \\ &\Leftrightarrow \exists x \in X : r \geq \phi(x, y) \\ &\Leftrightarrow \exists x \in X : (x, y, r) \in \phi \\ &\Leftrightarrow (y, r) \in \pi_{Y \times \mathbb{R}} \operatorname{epi} \phi. \end{aligned}$$

So, if ϕ is a convex function then $\operatorname{epi} \phi$ is a convex subset of $X \times Y \times \mathbb{R}$ which yields that $\mathfrak{C} = \pi_{Y \times \mathbb{R}} \operatorname{epi} \phi$ is convex, as well. □

The next proposition gives a presentation of the value of the problem (P) via its characterizing set $\mathfrak{C}$.

Proposition 3.3. *It holds*

$$v(\mathrm{P}) = \inf \overline{\{r \in \mathbb{R} : (0_Y, r) \in \mathfrak{C}\}}.$$

Proof. Let us denote $\mathcal{C} := \overline{\{r \in \mathbb{R} : (0_Y, r) \in \mathfrak{C}\}}$. We will prove that $v(\mathrm{P}) = \inf \mathcal{C}$.

Firstly, recall that

$$v(\mathrm{P}) = \inf_{x \in X} \phi(x, 0_Y). \tag{3.2}$$

Take arbitrarily $r \in \mathcal{C}$. Then, there exists a net $(r_i)_{i \in I}$ such that $(0_Y, r_i)_{i \in I} \subset \mathfrak{C}$ and $r_i \to r$. For each $i \in I$, as $(0_Y, r_i) \in \mathfrak{C}$, there is $x_i \in X$ such that $\phi(x_i, 0_Y) \leq r_i$. By (3.2), $\phi(x_i, 0_Y) \geq v(\mathrm{P})$, and hence, $v(\mathrm{P}) \leq r_i$ for all $i \in I$. Letting $r_i \to r$, we get $v(\mathrm{P}) \leq r$.

Take $\eta > v(\mathrm{P})$. It follows from (3.2) that there is $x_\eta \in X$ satisfying $\eta > \phi(x_\eta, 0_Y) := r_\eta$. Note that

$$(0_Y, r_\eta) \in \operatorname{epi} \phi(x_\eta, .) \subset \mathfrak{C},$$

which leads to $r_\eta \in \{r \in \mathbb{R} : (0_Y, r) \in \mathfrak{C}\} \subset \mathcal{C}$. Briefly, we have just shown that, for all $\eta > v(\mathrm{P})$, there exists $r_\eta \in \mathcal{C}$ such that that $\eta > r_\eta$.

So, $v(\mathrm{P}) = \inf \mathcal{C}$ and we are done. □

3.2. Lagrange dual problems

The *Lagrange dual problem* and the *loose Lagrange dual problem* of (P) are defined as follows, respectively,

$$(\mathrm{D}) \qquad \sup_{y^* \in Y^*} \inf_{(x,y) \in X \times Y} [\phi(x, y) + \langle y^*, y \rangle],$$

$$(\mathrm{D}_+) \qquad \sup_{y^* \in Y_+^*} \inf_{(x,y) \in X \times Y} [\phi(x, y) + \langle y^*, y \rangle].$$

It is worth noting that y^* in the dual problem (D) can be considered as the Lagrange multiplier while the one in (D_+) also can be understood as a positive Lagrange multiplier.

Proposition 3.4 (Weak duality)**.**

$$\nu(\mathrm{D}_+) \le \nu(\mathrm{D}) \le \nu(\mathrm{P}) < +\infty.$$

Proof. The first inequality follows immediately from the property of supremum. For the second inequality, taking arbitrarily $\bar{x} \in X$, we will prove that

$$\nu(\mathrm{D}) \le \phi(\bar{x}, 0_Y). \tag{3.3}$$

Indeed, for all $y^* \in Y^*$, one has

$$D_{y^*} := \inf_{(x,y)\in X\times Y} \{\phi(x,y) + \langle y^*, y\rangle\}$$
$$\le \phi(\bar{x}, 0_Y) + \langle y^*, 0_Y\rangle = \phi(\bar{x}, 0_Y).$$

Hence,

$$\nu(\mathrm{D}) = \sup_{y^*\in Y^*} D_{y^*} \le \phi(\bar{x}, 0_Y).$$

We have just shown that (3.3) holds for any $\bar{x} \in X$. This leads to the fact that

$$\nu(\mathrm{D}) \le \inf_{\bar{x}\in X} \phi(\bar{x}, 0_Y) = \nu(\mathrm{P}).$$

The last one comes from the fact that (P) is feasible, and the proof is complete. □

Theorem 3.1. *Assume that ϕ is convex. Then, one has*

$$\nu(\mathrm{D}) = \inf\{r \in \mathbb{R} : (0_Y, r) \in \overline{\mathfrak{C}}\}.$$

Moreover, if $\nu(\mathrm{D}) \in \mathbb{R}$ then

$$\nu(\mathrm{D}) = \min\{r \in \mathbb{R} : (0_Y, r) \in \overline{\mathfrak{C}}\}.$$

Proof. Denote

$$\mathcal{M} := \{r \in \mathbb{R} : (0_Y, r) \in \overline{\mathfrak{C}}\}.$$

It follows from Proposition 3.1 that $\mathcal{M} \neq \emptyset$. Let us divide the proof into three steps.

- Step 1. Take arbitrarily $r \in \mathcal{M}$. We claim that $\nu(\mathrm{D}) \le r$. As $r \in \mathcal{M}$, one has $(0_Y, r) \in \overline{\mathfrak{C}}$, and hence, there exists a net $((y_i, r_i))_{i\in I} \subset \mathfrak{C}$ such that $(y_i, r_i) \to (0_Y, r)$.

For each $i \in I$, as $(y_i, r_i) \in \mathfrak{C}$, there is $x_i \in X$ such that $(y_i, r_i) \in \operatorname{epi}\phi(x_i, .)$, or equivalently,

$$r_i \ge \phi(x_i, y_i). \tag{3.4}$$

Next, taking arbitrarily $y^* \in Y^*$, one has

$$\begin{aligned} D_{y^*} &:= \inf_{(x,y)\in X\times Y} \{\phi(x,y) + \langle y^*, y\rangle\} \\ &\le \phi(x_i, y_i) + \langle y^*, y_i\rangle. \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5) gives $D_{y^*} \le r_i + \langle y^*, y_i\rangle$ for all $i \in I$. Proceeding to the limit, we obtain $D_{y^*} \le r$ (recall that $(y_i, r_i) \to (0_Y, r)$). So,

$$\nu(\mathrm{D}) = \sup_{y^*\in Y^*} D_{y^*} \le r.$$

- Step 2. *Taking $\eta \in \mathbb{R}$ such that $\eta \ge \nu(\mathrm{D})$, we will show that $\eta \in \mathcal{M}$.* On the contrary, suppose that $\eta \notin \mathcal{M}$. Then, it follows from this that $(0_Y, \eta) \notin \overline{\mathfrak{C}}$. As ϕ is a convex function, the set $\mathfrak{C}$ is convex (see Proposition 3.2), and hence, $\overline{\mathfrak{C}}$ is convex as well. So, according to the separation theorem (see Rudin, 1991, Theorem 3.4), there are $\bar{y}^* \in Y^*$, $\bar{\alpha} \in \mathbb{R}$ and $\bar{\lambda} \in \mathbb{R}$ such that

$$\bar{\lambda}\eta < \bar{\alpha} < \langle \bar{y}^*, y\rangle + \bar{\lambda} r, \quad \forall (y,r) \in \mathfrak{C}. \tag{3.6}$$

We next prove that $\bar{\lambda} > 0$. Fix $\bar{x} \in \operatorname{dom}\phi(., 0_Y)$ (it is possible as $\operatorname{dom}(., 0_Y) \neq \emptyset$). Then we have $\phi(\bar{x}, 0_Y) \in \mathbb{R}$. Set $\bar{r} = \max\{\eta, \phi(\bar{x}, 0_Y)\}$. Then, one has $\bar{r} \ge \phi(\bar{x}, 0_Y)$, hence, $(0_Y, \bar{r}) \in \operatorname{epi}\phi(\bar{x}, .) \subset \mathfrak{C}$ which, together with (3.6), yields $\bar{\lambda}\eta < \bar{\lambda}\bar{r}$, or equivalently, $\bar{\lambda}(\bar{r} - \eta) > 0$. Combining this inequality with the fact that $\bar{r} \ge \eta$ (by the definition of $\bar{r}$) we obtain $\bar{\lambda} > 0$. Consequently, it follows from this and (3.6) that

$$\eta < \tilde{\alpha} < \langle \tilde{y}^*, y\rangle + r, \quad \forall (y,r) \in \mathfrak{C}, \tag{3.7}$$

where $\tilde{y}^* := \dfrac{1}{\bar{\lambda}} \bar{y}^*$ and $\tilde{\alpha} := \dfrac{1}{\bar{\lambda}} \bar{\alpha}$.

It is clear that for any $(x,y) \in \operatorname{dom}\phi$, one has

$$(y, \phi(x,y)) \in \operatorname{epi}\phi(x, .) \subset \mathfrak{C},$$

and hence, (3.7) entails

$$\eta < \tilde{\alpha} < \langle \tilde{y}^*, y\rangle + \phi(x,y).$$

Thus,

$$\eta < \tilde{\alpha} \le \inf_{(x,y)\in \mathrm{dom}\,\phi} \{\phi(x,y) + \langle \tilde{y}^*, y\rangle\}$$
$$= \inf_{(x,y)\in X\times Y} \{\phi(x,y) + \langle \tilde{y}^*, y\rangle\}.$$

This implies that

$$\eta < \sup_{y^*\in Y^*} \inf_{(x,y)\in X\times Y} \{\phi(x,y) + \langle y^*, y\rangle\} = \nu(\mathrm{D})$$

which contradicts the assumption $\eta \ge \nu(\mathrm{D})$. So, $\eta \in \mathcal{M}$ as desired.

• Step 3. *Conclusion*. We have just shown that:

(i) $\nu(\mathrm{D}) \le r$, $\forall r \in \mathcal{M}$ (Step 1).

(ii) Take $\zeta > \nu(\mathrm{D})$. Then, there exists $\eta \in \mathbb{R}$ such that $\zeta > \eta \ge \nu(\mathrm{D})$ (recall that $\nu(\mathrm{D}) < +\infty$, see Proposition 3.4). According to Step 2, one has $\eta \in \mathcal{M}$. Briefly, for all $\zeta > \nu(\mathrm{D})$, there is $\eta \in \mathcal{M}$ such that $\zeta > \eta$.

We thus get from (i) and (ii) that $\nu(\mathrm{D}) = \inf \mathcal{M}$.

We now assume further that $\nu(\mathrm{D}) \in \mathbb{R}$. Then, it is obvious that $\nu(\mathrm{D}) \ge \nu(\mathrm{D})$. Replacing η by $\nu(\mathrm{D})$ in Step 2, we get $\nu(\mathrm{D}) \in \mathcal{M}$. This, together with (i), yields that $\nu(\mathrm{D}) = \min \mathcal{M}$. □

Theorem 3.2. *Assume that* ϕ *is convex and the following condition holds*

(C_0) $\phi(\hat{x},.)$ *is bounded from above on* Y_+ *for some* $\hat{x} \in X$.

Then, $\nu(\mathrm{D}_+) = \inf\{r \in \mathbb{R} : (0_Y, r) \in \overline{\mathfrak{C}}\}$.

Moreover, if $\nu(\mathrm{D}_+) \in \mathbb{R}$, *then* $\nu(\mathrm{D}_+) = \min\{r \in \mathbb{R} : (0_Y, r) \in \overline{\mathfrak{C}}\}$.

Proof. Let us set

$$\mathcal{M} := \{r \in \mathbb{R} : (0_Y, r) \in \overline{\mathfrak{C}}\}.$$

It is easy to see that $\nu(\mathrm{D}_+) \le \nu(\mathrm{D})$. So, it follows from Theorem 3.1 that $\nu(\mathrm{D}_+) \le \inf \mathcal{M}$.

Next, taking $\eta \in \mathbb{R}$ such that $\eta \ge \nu(\mathrm{D}_+)$, we will show that $\eta \in \mathcal{M}$. Suppose, contrary to our claim, that $\eta \notin \mathcal{M}$. By the same argument as in Step 2 of the proof of the previous theorem, there exist $\tilde{y}^* \in Y^*$ and $\tilde{\alpha} \in \mathbb{R}$ such that

$$\eta < \tilde{\alpha} < \langle \tilde{y}^*, y\rangle + r, \quad \forall (y,r) \in \mathfrak{C}. \qquad (3.8)$$

• *We now prove that* $\tilde{y}^* \in Y_+^*$. To do this, take arbitrarily $\bar{k} \in Y_+$. Then, we only need to show that $\langle \tilde{y}^*, \bar{k}\rangle \ge 0$. As (C_0) holds, there are $\hat{x} \in X$ and $\hat{M} > 0$ such that $\phi(\hat{x}, k) \le \hat{M}$ for all $k \in Y_+$, which yields $\phi(\hat{x}, \mu\bar{k}) \le \hat{M}$ for all $\mu > 0$. Hence, for any $\mu > 0$, $(\mu\bar{k}, \hat{M}) \in \mathfrak{C}$, and then, (3.8) leads to

$$\eta < \langle \tilde{y}^*, \mu\bar{k}\rangle + \hat{M}, \quad \forall \mu > 0,$$

or equivalently,

$$\langle \tilde{y}^*, \bar{k}\rangle > \frac{\eta - \hat{M}}{\mu}, \quad \forall \mu > 0.$$

Letting $\mu \to +\infty$, one gets $\langle \tilde{y}^*, \bar{k}\rangle \ge 0$, which implies $\tilde{y}^* \in Y_+^*$.

• It is obvious that $(y, \phi(x,y)) \in \mathrm{epi}\,\phi(x,.) \subset \mathfrak{C}$ for all $(x,y) \in \mathrm{dom}\,\phi$. So, it follows from (3.8) that $\eta < \tilde{\alpha} < \langle \tilde{y}^*, y\rangle + \phi(x,y)$ for any $(x,y) \in \mathrm{dom}\phi$, and hence,

$$\eta < \tilde{\alpha} \le \inf_{(x,y)\in \mathrm{dom}\,\phi} \{\phi(x,y) + \langle \tilde{y}^*, y\rangle\}$$
$$= \inf_{(x,y)\in X\times Y} \{\phi(x,y) + \langle \tilde{y}^*, y\rangle\}$$
$$\le \sup_{y^*\in Y_+^*} \inf_{(x,y)\in X\times Y} \{\phi(x,y) + \langle y^*, y\rangle\}$$
$$= \nu(\mathrm{D}_+).$$

This contradicts our assumption $\eta \ge \nu(\mathrm{D}_+)$. Consequently, we arrive at $\eta \in \mathcal{M}$.

The rest of the proof runs as in Step 3 of the proof of Theorem 3.1, one gets $\nu(\mathrm{D}_+) = \inf \mathcal{M}$, and $\nu(\mathrm{D}_+) = \min \mathcal{M}$ if $\nu(\mathrm{D}_+) \in \mathbb{R}$. □

4. Characterization of zero duality gap under convex setting

We are in the position to establish the main results of this paper, that is characterizing zero duality gap for general vector optimization problem (P) in convex setting. We assume throughout this section that ϕ is a convex function.

Definition 4.1. We say that the problem (P) has *zero duality gap* if $\nu(\mathrm{P}) = \nu(\mathrm{D})$ and that (P) has *zero loose duality gap* if $\nu(\mathrm{P}) = \nu(\mathrm{D}_+)$.

According to Proposition 3.4, one has $\nu(\mathrm{D}_+) \le \nu(\mathrm{D}) \le \nu(\mathrm{P})$. So, if $\nu(\mathrm{P}) = \nu(\mathrm{D}_+)$ then $\nu(\mathrm{P}) = \nu(\mathrm{D})$, or in the other words, if (P) has zero loose duality gap then it has zero duality gap.

It is easy to see that

$$\overline{\mathfrak{C} \cap (\{0_Y\} \times \mathbb{R})} \subset \overline{\mathfrak{C}} \cap \overline{(\{0_Y\} \times \mathbb{R})} = \overline{\mathfrak{C}} \cap (\{0_Y\} \times \mathbb{R}), \quad (4.1)$$

where the last equality follows from the fact that $0_Y \times \mathbb{R}$ is a closed subset of $Y \times \mathbb{R}$. Let us introduce the *qualifying condition*:

$$(CQ) \qquad \overline{\mathfrak{C} \cap (\{0_Y\} \times \mathbb{R})} = \overline{\mathfrak{C}} \cap (\{0_Y\} \times \mathbb{R}), \quad (4.2)$$

which also means that the converse inclusion of (4.1) holds. It is observing that the condition (CQ) is a general type of the one introduced recently by Khanh *et al.* (2019) when they studied zero duality gap for linear programming problems.

Theorem 4.1 (Characterization of zero duality gap)**.** *The following statements are equivalent to each other:*

(i) (CQ) *holds.*

(ii) (P) *has zero duality gap.*

Proof. $[(i) \Rightarrow (ii)]$ Let $\pi_{\mathbb{R}} : Y \times \mathbb{R} \to \mathbb{R}$ be the conical projection from $Y \times \mathbb{R}$ to $\mathbb{R}$ (i.e., $\pi_{\mathbb{R}}(y,r) = r$ for all $(y,r) \in Y \times \mathbb{R}$). According to Proposition 3.3 and Theorem 3.1, one has

$$\nu(\mathrm{P}) = \inf \overline{\{r \in \mathbb{R} : (0_Y, r) \in \mathfrak{C}\}}$$
$$= \inf \pi_{\mathbb{R}} \left(\overline{\mathfrak{C} \cap \{0_Y\} \times \mathbb{R}}\right)$$

and

$$\nu(\mathrm{D}) = \inf \{r \in \mathbb{R} : (0_Y, r) \in \overline{\mathfrak{C}}\}$$
$$= \inf \pi_{\mathbb{R}} \left(\overline{\mathfrak{C}} \cap (\{0_Y\} \times \mathbb{R})\right).$$

So, if (CQ) holds then $\nu(\mathrm{P}) = \nu(\mathrm{D})$, which is nothing else but (ii).

$[(ii) \Rightarrow (i)]$ Assume that (ii) holds, i.e., $\nu(\mathrm{P}) = \nu(\mathrm{D})$. The proof is completed by showing that (i) holds. According to (4.1), it is sufficient to show that

$$\overline{\mathfrak{C}} \cap (\{0_Y\} \times \mathbb{R}) \subset \overline{\mathfrak{C} \cap (\{0_Y\} \times \mathbb{R})}. \quad (4.3)$$

For this purpose, we take $(0_Y, \bar{r}) \in \overline{\mathfrak{C}} \cap (\{0_Y\} \times \mathbb{R})$. We now show that $(0_Y, \bar{r}) \in \overline{\mathfrak{C} \cap (\{0_Y\} \times \mathbb{R})}$. Indeed, as

$$\nu(\mathrm{D}) = \inf \{r \in \mathbb{R} : (0_Y, r) \in \overline{\mathfrak{C}}\}$$

(see Theorem 3.1), one has $\bar{r} \ge \nu(\mathrm{D})$. Consequently, by assumption that (ii) holds, we obtain:

$$\bar{r} \ge \nu(\mathrm{P}) = \inf_{x \in X} \phi(x, 0_Y). \quad (4.4)$$

For each $n \in \mathbb{N}^*$, we set $r_n := \bar{r} + \frac{1}{n}$. The last inequality (4.4) implies that $r_n > \inf_{x \in X} \phi(x, 0_Y)$ for any $n \in \mathbb{N}^*$, which leads to the existence of $x_n \in X$ such that $r_n > \phi(x_n, 0_Y)$ for any $n \in \mathbb{N}^*$. Hence, $(0_Y, r_n) \in epi\phi(x_n, .) \subset \mathfrak{C}$, giving rise to $(0_Y, r_n) \in \mathfrak{C} \cap (0_Y \times \mathbb{R})$. This, together with the fact that $(0_Y, r_n) \to (0_Y, \bar{r})$, yields $(0_Y, \bar{r}) \in \overline{\mathfrak{C} \cap (\{0_Y\} \times \mathbb{R})}$, which completes the proof. □

Example 4.1. Let X_+ be a non-empty convex cone in X. We consider the *equality constrained linear programming problem* of the form:

$$(\mathrm{EP}) \qquad \inf \langle \zeta, x \rangle$$
$$\text{s.t. } Ax = b$$
$$x \in X_+$$

where $\zeta \in X^*$, $b \in Y$, and A being a continuous linear function from X to Y.

Let us introduce the perturbation mapping $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ such that

$$\phi(x,y) = \begin{cases} \langle \zeta, x \rangle & \text{if } Ax + y = b \text{ and } x \in X_+ \\ +\infty & \text{else.} \end{cases}$$

Then, (EP) can be rewritten as $\inf_{x \in X} \phi(x, 0_Y)$ in the form of (P). The characterizing set $\mathfrak{C}$ now reduces to the set

$$M = \left\{\left(b - Ax, \langle \zeta, x \rangle + r\right) : x \in X_+, r \ge 0\right\}$$

while the dual problem (D) becomes

$$(\mathrm{ED}) \qquad \sup \langle y^*, b \rangle$$
$$\text{s.t. } \zeta - A^\# y^* \in X_+^*$$
$$y^* \in Y^*.$$

In this case, the condition (CQ) *collapses to* $\overline{M}\cap(\{0_Y\}\times\mathbb{R})=\overline{M\cap(\{0_Y\}\times\mathbb{R})}$. According to Theorem 4.1, one has $\inf(\mathrm{EP})=\sup(\mathrm{ED})$ if and only if $\overline{M}\cap(\{0_Y\}\times\mathbb{R})\subset\overline{M\cap(\{0_Y\}\times\mathbb{R})}$.

Theorem 4.2 (Characterization of zero loose duality gap)**.** *Assume that the condition* (C_0) *in Theorem 3.2 is fulfilled. Then, the following statements are equivalent to each other:*

(i) (CQ) *holds.*

(ii) (P) *has zero loose duality gap.*

Proof. Similar to the proof of Theorem 4.1, using Theorem 3.2 instead of Theorem 3.1. □

We now consider the new qualifying condition

$$(CQR)\qquad \mathfrak{C}\cap(\{0_Y\}\times\mathbb{R})=\overline{\mathfrak{C}}\cap(\{0_Y\}\times\mathbb{R}).$$

We say that $\mathfrak{C}$ is closed regarding the set $0_Y\times\mathbb{R}$ if (CQR) holds. It is worth observing that if (CQR) holds, then (CQR) does, too.

The next corollary is an immediate consequence of the above theorems.

Corollary 4.1 *Assume that* (CQR) *holds. Then, it holds:*

(i) (P) *has zero duality gap.*

(ii) *If* (C_0) *in Theorem 3.2 holds then* (P) *has zero loose duality gap.*

Proof. As (CQR) holds, one has

$$\overline{\mathfrak{C}\cap(\{0_Y\}\times\mathbb{R})}=\overline{\overline{\mathfrak{C}}\cap(\{0_Y\}\times\mathbb{R})}=\overline{\mathfrak{C}}\cap(\{0_Y\}\times\mathbb{R}),$$

which means that (CQ) holds. The conclusion now follows from Theorems 4.1 and 4.2. □

5. Application: Zero duality gap for composite optimization problems

In this last section, we apply the general results established in the previous sections to derive zero duality gap for the composite optimization problem. We are concerned with the *composite optimization problems,* of the form (Boţ, 2010, Boţ *et al.*, 2005, Dinh and Mo, 2012)

$$(\mathrm{CP})\qquad \inf_{x\in X}[f(x)+(g\circ H)(x)],$$

where X, Z are locally convex Hausdorff topological vector spaces, Z_+ is non-empty convex cone in Z, $f:X\to\overline{\mathbb{R}}$, $g:Z\to\overline{\mathbb{R}}$, and $H:X\to Z^\bullet$ are proper mappings such that $\mathrm{dom}\, f\cap H^{-1}(\mathrm{dom}\, g)\neq\emptyset$ and we adopt the convention $g(+\infty_Z)=+\infty$.

In the rest of this section, we will establish various characterizations of zero duality gap for the problem (CP) due to different choices of the perturbation function ϕ introduced in Section 1.

5.1. The first way of transforming

Consider $Y=Z$, $Y_+=Z_+$, and $\phi_1:X\times Z\to\overline{\mathbb{R}}$ defined by

$$\phi_1(x,z)=f(x)+g(H(x)-z).\qquad(5.1)$$

It is easy to see that

$$\mathrm{dom}\,\phi_1(.,0_Z)=\mathrm{dom}(f+g\circ H)$$
$$=\mathrm{dom}\, f\cap H^{-1}(\mathrm{dom}\, g),$$

and hence, by above assumption, $\mathrm{dom}\,\phi_1(.,0_Z)\neq\emptyset$.

It is worth noting that when taking $\phi=\phi_1$, the problem (P) collapses to the problem (CP). In this case, characterizations of zero duality gap for the problem (P) are also the ones for the problem (CP).

The next lemma gives us specific forms of the characterization set $\mathfrak{C}$ and dual problems (D) and (D_+) in this setting.

Lemma 5.1. *With* $Y=Z$, $Y_+=Z_+$, *and* $\phi=\phi_1$ *given by (5.1), the set* $\mathfrak{C}$, *the problems* (D) *and* (D_+) *become, respectively,*

$$\mathfrak{C}_1:=\mathrm{im}(H,f)-\mathrm{hyp}(-g),\qquad(5.2)$$

and

$$(\mathrm{C}D^1)\quad \sup_{z^*\in\mathrm{dom}\, g^*}\{-g^*(z^*)+\inf_{x\in X}[f(x)+\langle z^*,H(x)\rangle]\},$$

$$(\mathrm{C}D^1_+)\quad \sup_{z^*\in\mathrm{dom}\, g^*\cap Z^*_+}\{-g^*(z^*)+\inf_{x\in X}[f(x)+\langle z^*,H(x)\rangle]\},$$

where $\mathrm{im}(H,f)=\{(H(x),f(x)):x\in\mathrm{dom}\, H\cap\mathrm{dom}\, f\}$.

Proof. See Appendix A. □

We now establish the first characterization of zero duality gap for the problem (CP) and the one of zero loose duality gap for the problem (CP).

Corollary 5.1 (Characterization of zero duality gap 1)**.** *Assume that* f *is convex, that* g *is convex and* Y_+ *-nondecreasing, and that* H *is a* Y_+ *-convex mapping. Then, the following statements are equivalent:*

(*i*) $\overline{\mathfrak{C}_1 \cap (\{0_Z\} \times \mathbb{R})} = \overline{\mathfrak{C}_1} \cap (\{0_Z\} \times \mathbb{R})$,

(*ii*) $\nu(\mathrm{C}P) = \nu(\mathrm{C}D^1)$.

Proof. The convexity of ϕ_1 implies directly from the above assumption. Then, the conclusion follows from Theorem 4.1 and Lemma 5.1. □

Corollary 5.2 (Characterization of zero loose duality gap 1)**.** *Assume that the assumption of Corollary 5.1 holds. Assume further that the following condition holds*

(C_1) $\exists \hat{x} \in X : g(H(\hat{x}) - Z_+)$ *is bounded from above.*

Then, the following statements are equivalent:

(*i*) $\overline{\mathfrak{C}_1 \cap (\{0_Z\} \times \mathbb{R})} = \overline{\mathfrak{C}_1} \cap (\{0_Z\} \times \mathbb{R})$,

(*ii*) $\nu(\mathrm{C}P) = \nu(\mathrm{C}D^1_+)$.

Proof. It follows from Theorem 4.2 and Lemma 5.1. □

5.2. The second way of transforming

We now take $Y = X \times Z, Y_+ = \{0_X\} \times Z_+$, and the perturbation $\phi_2 : X \times X \times Z \to \overline{\mathbb{R}}$ defined by

$$\phi_2(x, x', z) = f(x + x') + g(H(x) - z). \tag{5.3}$$

It is easy to check that $\operatorname{dom}\phi_2(., 0_X, 0_Z) \neq \varnothing$. It is worth observing that in this case, taking $\phi = \phi_2$, the problem (P) collapses to the problem (CP).

The formulas of characterization set $\mathfrak{C}$ and dual problems (D) and (D_+) in this the case are given by the following lemma.

Lemma 5.2. *With* $Y = X \times Z$, $Y_+ = \{0_X\} \times Z_+$ *and* $\phi = \phi_2$ *given by (5.3), the set* $\mathfrak{C}$ *becomes*

$$\mathfrak{C}_2 := \operatorname{gr}(0_Z, f) - \operatorname{gr}(-H, 0) - \{0_X\} \times \operatorname{hyp}(-g), \tag{5.4}$$

while the problems (D) *and* (D_+) *become, respectively,*

$$(\mathrm{C}D^2) \quad \sup_{(x^*, z^*) \in \operatorname{dom} f^* \times \operatorname{dom} g^*} \{-f^*(x^*) - g^*(z^*) - (z^* \circ H)^*(-x^*)\},$$

$$(\mathrm{C}D^2_+) \quad \sup_{\substack{x^* \operatorname{dom} f^* \\ z^* \in \operatorname{dom} g^* \cap Y_+^*}} \{-f^*(x^*) - g^*(z^*) - (z^* \circ H)^*(-x^*)\}.$$

Proof. See Appendix B. □

By combining Lemma 5.2 to Theorem 4.1 and to Theorem 4.2, respectively, we get directly the consequences as follows:

Corollary 5.3 (Characterization of zero duality gap 2)**.** *Assume all the assumption of Corollary 5.1. Then, the following statements are equivalent:*

(*i*) $\overline{\mathfrak{C}_2 \cap (\{0_X\} \times \mathbb{R} \times \{0_Z\})} = \overline{\mathfrak{C}_2} \cap (\{0_X\} \times \mathbb{R} \times \{0_Z\})$,

(*ii*) $\nu(\mathrm{C}P) = \nu(\mathrm{C}D^2)$.

Corollary 5.4 (Characterization of zero loose duality gap 2)**.** *Assume all the assumption of Corollary 5.1. Assume further that the condition* (C_1) *in Corollary 5.2 holds Then, the following statements are equivalent:*

(*i*) $\overline{\mathfrak{C}_2 \cap (\{0_X\} \times \mathbb{R} \times \{0_Z\})} = \overline{\mathfrak{C}_2} \cap (\{0_X\} \times \mathbb{R} \times \{0_Z\})$,

(*ii*) $\nu(\mathrm{C}P) = \nu(\mathrm{C}D^2_+)$.

References

Anderson, E.J. (1983). A review of duality theory for linear programming over topological vector spaces. *J. Math. Anal. Appl.*, *97*(2), 380-392.

Boţ, R.I. (2010). *Conjugate Duality in Convex Optimization*. Berlin: Springer.

Boţ, R.I., Hodrea, I. B., and Wanka, G. (2005). Composed convex programming: duality and Farkas-type results. *Proceeding of the International Conference In Memoriam Gyula Farkas*, 23-26.

Feizollahi, M.J., Ahmed,. S., and Sun, A. (2017). Exact augmented Lagrangian duality for mixed integer linear programming. *Math. Program.,161*, 365-387.

Huang, X.X. and Yang, X.Q. (2003). A unified augmented Lagrangian approach to duality and exact penalization. *Math. Oper. Res.*, *3*, 533-552.

Huang, X.X. and Yang, X.Q. (2005). Further study on augmented Lagrangian duality theory.*J. Global Optim.*, *31*, 193-210.

Jeyakumar, V. and Li, G.Y. (2009a). Stable zero duality gaps in convex programming:

Complete dual characterizations with application to semidefinite programs, *J. Math. Anal. Appl.*, *360*, 156-167.

Jeyakumar, V. and Li, G.Y. (2009b). New dual constraint qualifications characterizing zero duality gaps of convex programs and semidefinite programs. *Nonlinear Anal., 71*, 2239-2249.

Jeyakumar, V. and Wolkowicz, H. (1990). Zero duality gaps in infinite-dimensional programming. *J. Optim. Theory Appl.*, *67*(1), 87-108.

Li, D. (1995). Zero duality gap for a class of nonconvex optimization problems. *J. Optim. Theory Appl.*, *85*(2), 309-324.

Li, D. (1999). Zero duality gap in integer programming: P-norm surrogate constraint method. *Oper. Res. Lett.*, *25*(2), 89-96.

Long, F. and Zeng, B. (2020). The zero duality gap property for an optimal control problem governed by a multivalued hemivariational inequality. *Applied Mathematics & Optimization*, 10.1007/s00245-020-09721-z.

Nguyen Dinh, Dang Hai Long, Tran Hong Mo, and Yao, J.-C. (2020). Approximate Farkas lemmas for vector systems with applications to convex vector optimization problems. *J. Nonlinear Convex Anal.*, *21*(5), 1225-1246.

Nguyen Dinh and Tran Hong Mo. (2012). Qualification conditions and Farkas-type results for systems involving composite functions.*Vietnam J. Math.*, *40*(4), 407-437.

Pham Duy Khanh, Tran Hong Mo, and Tran Thi Tu Trinh. (2019). Necessary and sufficient conditions for qualitative properties of infinite dimensional linear programming problems. *Numer. Func. Anal. Opt.*, *40*, 924-943.

Rubinov, A.M., Huang, X.X., and Yang, X.Q. (2002). The zero duality gap property and lower semicontinuity of the perturbation function. *Math. Oper. Res.*, *27*(4), 775-791.

Rudin, W. (1991). *Functional Analysis (2nd Edition)*. New York: McGraw-Hill.

Yang, X.Q. and Huang, X.X. (2001). A nonlinear Lagrangian approach to constrained optimization problems. *SIAM J. Optim.*, *11*(4), 1119-1144.

Appendix

Proof of Lemma 5.1.

(i) *Prove that* $\mathfrak{C} = \mathfrak{C}_1$. Take $(z,r) \in \mathfrak{C}$. Then, there exists $x \in X$ such that $(z,r) \in \operatorname{epi}\phi_1(x,.)$, which means $r \geq \phi_1(x,z) = f(x) + g(H(x) - z)$, or equivalently, $f(x) - r \leq -g(H(x) - z)$. So, $(H(x) - z, f(x) - r) \in \operatorname{hyp}(-g)$, and hence,

$$\begin{aligned}(z,r) &= (H(x), f(x)) - (H(x) - z, f(x) - r)\\ &\in (H(x), f(x)) - \operatorname{hyp}(-g).\end{aligned}$$

Moreover, the inequality $r \geq f(x) + g(H(x) - z)$ also leads to $x \in \operatorname{dom} f \cap \operatorname{dom} H = \operatorname{dom}(H, f)$. So, one gets $(z,r) \in \operatorname{im}(H, f) - \operatorname{hyp}(-g)$.

Take $(z,r) \in \mathfrak{C}_1$. Then, there are $x \in \operatorname{dom}(H, f) = \operatorname{dom} f \cap \operatorname{dom} H$ and $(u,\alpha) \in \operatorname{hyp}(-g)$ such that $(z,r) = (H(x), f(x)) - (u,\alpha)$, which means

$$z = H(x) - u \quad \text{and} \quad r = f(x) - \alpha. \tag{5.5}$$

As $(u,\alpha) \in \operatorname{hyp}(-g)$, one has $\alpha \leq -g(u)$, or equivalently, $-\alpha \geq g(u)$, and hence, by (5.5), $r \geq f(x) + g(u) = f(x) + g(H(x) - z) = \phi_1(x,z)$. This yields $(z,r) \in \operatorname{epi}\phi_1(x,.) \subset \mathfrak{C}_1$ and we are done.

(ii) *Prove that* $\sup(\mathrm{D}) = \sup(\mathrm{C}D^1)$. By the definition of the Lagrange dual problem (D) (see Subsection 3.2), one has $\sup(\mathrm{D}) = \sup_{z^* \in Z^*} D_{z^*}$ where

$$D_{z^*} := \inf_{(x,z)\in X\times Z} [\phi_1(x,z) + \langle z^*, z\rangle]$$

(recall that, at this time, $Y = Z$ and $\phi = \phi_1$).

For each $z^* \in Z^*$, according to (5.1), we have

$$\begin{aligned}D_{z^*} &= \inf_{(x,z)\in X\times Z} [f(x) + g(H(x) - z) + \langle z^*, z\rangle]\\ &= \inf_{(x,u)\in X\times Z} [f(x) + g(u) + \langle z^*, H(x) - u\rangle]\\ &= -\sup_{u\in Z}[\langle z^*, u\rangle - g(u)] + \inf_{x\in X}[f(x) + \langle z^*, H(x)\rangle]\\ &= -g^*(z^*) + \inf_{x\in X}[f(x) + \langle z^*, H(x)\rangle].\end{aligned}$$

So, we get

$$\sup(\mathrm{D}) = \sup_{z^* \in Z^*} \{-g^*(z^*) + \inf_{x\in X}[f(x) + \langle z^*, H(x)\rangle]\}$$
$$= \sup_{z^* \in \mathrm{dom}\, g^*} \{-g^*(z^*) + \inf_{x\in X}[f(x) + \langle z^*, H(x)\rangle]\}$$
$$= \sup(\mathrm{C}D^1)$$

where the third equality follows from the fact that $g^*(u^*) = +\infty$ whenever $u^* \notin \mathrm{dom}\, g^*$.

(iii) Similar arguments apply to the problem (D_+) to obtain $\sup(\mathrm{D}_+) = \sup(\mathrm{C}D^1_+)$, and the proof is complete. □

Proof of Lemma 5.2.

Prove that $\mathfrak{C} = \mathfrak{C}_2$. Take $(x', z, r) \in \mathfrak{C}$. Then, there is $x \in X$ such that $(x, x', z, r) \in \mathrm{epi}\, \phi_2(x,.,.)$, i.e.,

$$r \geq \phi_2(x, x', z) = f(x + x') + g(H(x) - z). \quad (5.6)$$

On the other hand, we can rewrite (x', z, r) as

$$(x', z, r) = (x + x', 0_Z, f(x + x')) - (x, -H(x), 0)$$
$$-(0_X, H(x) - z, f(x + x') - r). \qquad (5.7)$$

It follows from (5.6) that

$$x + x' \in \mathrm{dom}\, f = \mathrm{dom}(0_Z, f),$$
$$x \in \mathrm{dom}\, H = \mathrm{dom}(-H, 0),$$

and

$$f(x + x') - z \leq -g(H(x) - z).$$

This, together with (5.6), yields

$(x', z, r) \in \mathrm{gr}(0_Z, f) - \mathrm{gr}(-H, 0) - \{0_X\} \times \mathrm{hyp}(-g)$.

Conversely, take $(x', z, r) \in \mathfrak{C}_2$. Then, there are

$u \in \mathrm{dom}(0_Z, f) = \mathrm{dom}\, f$, $v \in \mathrm{dom}(-H, 0) = \mathrm{dom}\, H$,

and

$(w, \alpha) \in \mathrm{hyp}(-g)$

such that

$(x', z, r) = (u, 0_Z, f(u)) - (v, -H(v), 0) - (0_X, w, \alpha)$,

and hence

$$x' = u - v, \quad z = H(v) - w \quad \text{and} \quad r = f(u) - \alpha. \quad (5.8)$$

As $(w, \alpha) \in \mathrm{hyp}(-g)$, we have $\alpha \leq -g(u)$. Combining this with (5.8) $-\alpha \geq g(u)$, we get

$$r \geq f(u) + g(w) = f(v + x') + g(H(v) - z)$$
$$= \phi_2(u, x', z).$$

Consequently, $(x', z, r) \in \mathrm{epi}\, \phi_2(u,.) \subset \mathfrak{C}_2$ and we are done.

The proof of equalities $\sup(\mathrm{D}) = \sup(\mathrm{C}D^2)$ and $\sup(\mathrm{D}) = \sup(\mathrm{C}D^2_+)$ is similar as in that of Lemma 5.1. □