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Transition nodal basis functions in p-adaptive finte element methods

Hàm nút cơ sở chuyển giao dùng trong phương pháp phần tử hữu hạn thích nghi loại *p*

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Abstract

In this paper, we present two different approaches to define transition nodal basis functions that can be used to allow elements of different degrees on the same mesh in p-adaptive finite element method.

Keywords: Nodal basis functions; Transition nodal basis functions; *p*-adaptive finite elements.

Tóm tắt

Trong bài báo này, chúng tôi giới thiệu hai cách xây dựng hàm nút cơ sở chuyển giao dùng trong phương pháp phần tử hữu hạn thích nghi loại *p*. Các cách xây dựng này có thể cho phép các phần tử có bậc khác nhau cùng tồn tại trong một lưới tính toán.

Từ khóa: Điểm nút; Hàm nút cơ sở; Phần tử hữu hạn loại *p*.

1. Introduction

Finite element is one of the most popular methods to solve partial differential equations arising from both academic and engineering problems. In [1], the authors have introduced nodal points, nodal basis fuctions used in the *p*-version of finite element methods (see [2, 3, 4]), where elements have the same degree. In this paper, we will present different approaches to formulate transition nodal basis functions so that elements of different degrees can co-exist in the same mesh (triangulation). This flexibility helps *p*-adaptive finite element methods to better capture the behavior of the exact solution by adaptively choosing degrees for elements (see [3, 5, 6, 7, 8, 9, 10, 11, 12]).

In a mesh with varying degrees, along the interfaces separating elements of different degrees, it is natural to use nodal points of higher degree for shared edges. Therefore, along degree interfaces, only elements of lower degrees need new sets of basis functions. These elements are called *transition elements*.

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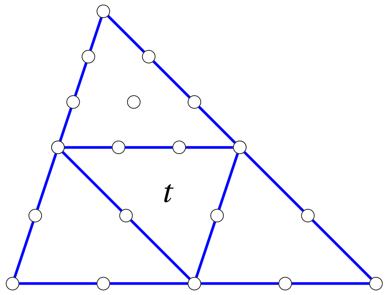


Figure 1. A transition element (t in the middle).

2. The first approach

Consider an admissible mesh, where there is no violation of 1-irregular rule and 2-neighbor rule, (for more details on these rules, see [13, 14]). In this mesh, a *transition element* is an element of degree p having one and just one neighbor of degree $p+1$ (the other neighbors if exist are of degree p). Figure 1 illustrates a transition element t with its neighbors. The edge shared by t and its neighbor of higher degree $(p+1)$ is called *transition edge*.

To define a set of *transition basis functions* for t, we will follow the same idea of [1] by making each basis function equal 1 at one nodal point and equal 0 at all other nodal points.

Assume edge two is the transition edge of t. By [1, Corollary 3.4], all of the basis functions ϕ_j in $\mathscr{P}_p(t)$ that are not associated with edge two equal zero on the whole of edge two. In particular, these functions equal zero at nodal points of degree $p+1$ on the transition edge. Therefore, we can use these functions in the set of basis functions for transition element t without modification.

It now remains to define basis functions associated with nodal points of degree $p+1$ on the transition edge (including the two vertex nodal points). Label these points $n_{v_1}, n_{v_3}, n_{e_1}, \ldots, n_{e_{p-1}}$, where the first two are vertex nodal points and the rest are edge nodal points. Denote θ_j the function of the straight line perpendicular to edge two at n_{e_j} and S_{int} the set of interior nodal points of t. Let

$$\psi^{(e_i)} = C_i c_1 c_3 \prod_{\substack{j=1\\ j\neq i}}^{p-1} \theta_j - \sum_{j\in S_{int}} \alpha_{ij}\phi_j. \qquad (1)$$

Here c_1 and c_3 are equations of edge one and edge three; C_i and α_{ij} are chosen so that $\psi^{(e_i)}$ equals 1 at n_{e_i} and equals 0 at all the other nodal points of degree $p+1$ on edge two, as well as all interior nodal point of t. Clearly, $\psi^{(e_i)}$ also equals zero at nodal points of degree p on edge one and edge three. Hence $\psi^{(e_i)}$ are the transition edge basis functions for t.

Now we define the basis functions associated with the two vertices on the transition edge. We begin with standard vertex basis functions and use $\psi^{(e_i)}$ to modify them to have right values on the transition edge.

$$\psi^{(v_i)} = \phi_{v_i} - \sum_{j=1}^{p-1} \beta_{ij}\psi^{(e_j)}, \qquad (2)$$

where $i = 1,3$ and b_{ij} are chosen such that $\psi^{(v_i)}$ equals zero at all edge nodal points of degree $p+1$ on the transition edge. Obviously, $\psi^{(v_i)}$ equals 1 at the vertex v_i.

In summary, with two vertex basis functions defined by equation (2), $p-1$ edge basis functions defined by equation (1), and standard nodal basis functions of degree p not associated the transition edge, we have a set of N_p+1 functions that equal 1 at one nodal point and equal 0 at all other nodal points of t. An argument similar to the one in [1, Theorem 3.3] shows that the set is linearly independent. Let $\mathscr{P}_{p+1/2}(t)$ be the space spanned by that set of basis functions. Then $\mathscr{P}_{p+1/2}(t)$ is a polynomial space for the transition element t. Naturally we would want this newly defined space to contain the regular space of polynomials of degree p restricted on t. The following theorem ensures that desire.

Theorem 2.1. *$\mathscr{P}_p(t)$ is a subset of $\mathscr{P}_{p+1/2}(t)$.*

Proof. Since $\mathscr{P}_{p+1/2}(t)$ includes all of the basis functions in $\mathscr{P}_p(t)$ that are not associated with nodal point on the transition edge, it suffices to show that basis functions of degree p associated with the transition edge is contained in $\mathscr{P}_{p+1/2}(t)$.

From equation (2), we can write the standard vertex basis functions ϕ_{v_i} as a linear combination of functions in $\mathscr{P}_{p+1/2}(t)$:

$$\phi_{v_i} = \psi^{(v_i)} + \sum_{j=1}^{p-1} \beta_{ij}\psi^{(e_j)}, \text{ for } i = 1,3.$$

Therefore, ϕ_{v_1} and ϕ_{v_3} are contained in $\mathscr{P}_{p+1/2}(t)$.

Now we show that the standard basis functions of degree p are also a linear combination of functions in $\mathscr{P}_{p+1/2}(t)$. Denote $\hat{\theta}_j$ the function of straight line perpendicular to transition edge at $n_{\hat{e}_j}$, a nodal point of degree p. Let

$$\hat{\psi}^{(\hat{e}_i)} = \hat{C}_i c_1 c_3 \prod_{\substack{j=1\\ j\neq i}}^{p-2} \hat{\theta}_j - \sum_{j\in S_{int}} \hat{\alpha}_{ij}\phi_j$$

where $\hat{C}_i$ and $\hat{\alpha}_{ij}$ are chosen so that $\hat{\psi}^{(\hat{e}_i)}$ equals 1 at $\hat{e}_i$ and equals 0 at all of the other nodal points of degree p of t. By the uniqueness of basis functions proved in [1, Theorem 3.3], $\hat{\psi}^{(\hat{e}_i)}$ is actually the nodal basis function of $\mathscr{P}_p(t)$ associated with the nodal point $\hat{e}_i$. Because $\phi_j \in \mathscr{P}_{p+1/2}(t)$ for $j \in S_{int}$, it is now sufficient to show that $\{\hat{\Theta}_j\}_{j=1}^{p-2}$ can be written as linear combinations of $\{\Theta_j\}_{j=1}^{p-1}$, where

$$\hat{\Theta}_i = \prod_{\substack{j=1\\ j\neq i}}^{p-2} \hat{\theta}_j \quad \text{and} \quad \Theta_i = \prod_{\substack{j=1\\ j\neq i}}^{p-1} \theta_j.$$

Since θ_j and $\hat{\theta}_j$ are lines of the same direction, the problem is reduced to one dimensional case: show that polynomials of degree $p-3$ can be written as linear combinations of basis polynomials of degree $p-2$. This statement is obviously true. □

Remark 2.2. *The set of basis functions defined above is not a unique one. For a transition element, there are more than one set of basis functions that equal 1 at a nodal point and equal 0 at all of the others.*

3. The second approach

In the previous subsection, we considered only meshes with no violation of 1-irregular rule and 2-neighbors rule. Now we consider more general meshes that might have violations of those two rules. In these meshes, a *transition element* is an element of degree p with at least one of its neighbors of degree $p+1$ or higher.

For the sake of clarity, we begin with a transition element t of degree p having one neighbor of degree $p+1$ and no other neighbor of degree higher than p. Without loss of generality, we can assume that the higher degree neighbor is across edge three of t. In other words, edge three is a *transition edge* of t.

Similar to the previous subsection, we can use the standard basis functions *basis functions! transition basis functions* of degree p at the nodal points that are not associated with edge three. Again, it remains to define the basis functions associated with the transition edge three.

Define a special polynomial of degree $p+1$, which is zero at all standard nodal points of degree p of t, and identically zero on edges one and two by

$$\tilde{\phi}_{(p+1)} = \prod_{k=0}^{(p-1)/2} (c_1 - k/p)(c_2 - k/p)$$

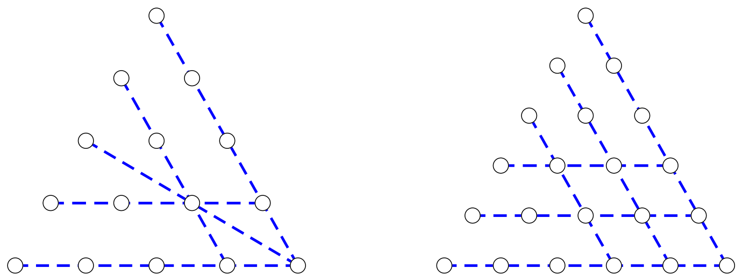
for odd p, and

$$\tilde{\phi}_{(p+1)} = (c_1 - c_2) \prod_{k=0}^{(p-2)/2} (c_1 - k/p)(c_2 - k/p)$$

for even p.

This polynomial is actually a product of equal number of lines parallel to edge one and edge two for p is odd, and is that same product multiplied with the median from vertex three for p is even. Figure 2 represents the lines in the formula of $\tilde{\phi}_{(p+1)}$ for $p = 4,5$.

A polynomial space for the transition element is given by $\tilde{\mathscr{P}}(t) = \mathscr{P}_p(t) \oplus \tilde{\phi}_{(p+1)}$. In other words, we form $p+2$ basis functions $\{\psi_i\}_{i=1}^{p+2}$ as linear combinations of $\tilde{\phi}_{(p+1)}$ and N_p standard basis



(a) $p=4$ (b) $p=5$

Figure 2. Lines in the formula of $\tilde{\phi}_{p+1}$ for $p=4,5$.

functions of degree p of t. Here ψ_i is the basis function associated with the nodal point n_i on the transition edge. Denote S and S_{trans} the set of nodal points of t and that associated with the transition edge, respectively. We have

$$\psi_i = \sum_{j\in S} \alpha_{ij}\phi_j + c_i\tilde{\phi}_{p+1}. \quad (3)$$

Matching both sides of equation (3) above at standard nodal points of t that are not associated with the transition edge yields $\alpha_{ij}=0$ for $j \notin S_{trans}$. Therefore equation (3) becomes

$$\psi_i = \sum_{j\in S_{trans}} \alpha_{ij}\phi_j + c_i\tilde{\phi}_{p+1}. \quad (4)$$

This equation implies that ψ_i can actually be written as a linear combination of $\tilde{\phi}_{(p+1)}$ and $p+1$ standard basis functions of degree p associated with the transition edge. Since we know values of ψ_i at $p+2$ points on edge three, coefficients α_{ij} and c_i in equation (4) can be determined by solving a $(p+2)\times(p+2)$ system of linear equations. This approach is considered expensive as we have to solve a system of size $p+2$ for each basis function ϕ_i.

In PLTMG [15], the coefficients α_{ij} are computed by matching both sides of equation (4) at standard nodal points of degree p associated with the transition edges. At each point, $\tilde{\phi}_{p+1}$ equals 0 and all of the ϕ_j equal 0 except one equals 1. As of ψ_i, we do not know its complete formula but its values on the transition edge are defined by its $p+2$ values at nodal points of degree $p+1$. In addition, the coefficient c_i can be computed by taking $(p+1)$th derivative of equation (4) in the tangential direction for the transition edge. In this approach of computation, all the coefficients are geometry independent. Therefore, we only need to do the calculation once and use the results for all elements.

Now we consider a more general case, where the higher degree element is of degree $p+k$, for $k>1$. Similarly, we can define a polynomial space for t as

$$\tilde{\mathscr{P}}(t) = \mathscr{P}_p(t) \oplus \{\tilde{\phi}_{(p+1)}(c_1-c_2)^m\}_{m=0}^{k-1},$$

and the transition basis function ψ_i is given by

$$\psi_i = \sum_{j\in S_{trans}} \alpha_{ij}\phi_j + \sum_{m=0}^{k-1} c_{i,m}\tilde{\phi}_{p+1}(c_1-c_2)^m. \quad (5)$$

Here the coefficients $c_{i,m}$ can be consecutively computed by taking $(p+m+1)$st derivative of equation (5), and α_{ij} are computed as in the previous case.

In this approach, we can be even more general by allowing one element to have more than one transition edge. The transition basis functions associated with transition edges are defined consecutively and almost independently. Assume edge two is the only transition edge left. Similarly, we can define a polynomial space for t as

$$\tilde{\mathscr{P}}(t) = \mathscr{P}_p(t) \oplus \{\tilde{\phi}^{(3)}_{(p+1)}(c_1-c_2)^m\}_{m=0}^{k^{(3)}-1} \oplus \{\tilde{\phi}^{(2)}_{(p+1)}(c_3-c_1)^m\}_{m=0}^{k^{(2)}-1}.$$

After defining the transition basis functions associated with edge three we can define those associated with edge two as in equation (5). The only difference is that the basis function ϕ_j associated with vertex one, is now ψ_{v1}, the transition basis function associated with edge three at vertex v_1.

If a third transition edge is present, it is treated analogously.

Theorem 3.1. *The finite element spaces constructed in the two approaches are C^0.*

Proof. Let e be the shared edge of two elements t and t', where t is of degree p and t' is of degree $p+k$, $k \geq 1$. Assume $P \in \tilde{\mathscr{P}}(t)$ and $Q \in \tilde{\mathscr{P}}(t')$ (if t' is not a transition element we can think of $\tilde{\mathscr{P}}(t')$ as $\mathscr{P}_{p+k}(t')$) agree at $p+k+1$ nodal points of degree $p+k$ on e. We will show that P and Q agree along the whole edge e.

Clearly, $\tilde{\mathscr{P}}(t)$ and $\tilde{\mathscr{P}}(t')$ could contain polynomials of degree higher than $p+k$, namely transition basis functions associated with edges other than e. However, in both approaches 2 and 3, these basis functions are defined to equal 0 on the whole e. Therefore, the restrictions of P and Q on e are 1-dimensional polynomials of degree $p+k$ or less.

An argument similar to the one [1, Proposition 3.5] shows that P and Q agree on the whole edge e. □

4. Conclusion

In this paper, we have formulated two different approaches to define transition nodal basis functions that can be used in p-adaptive finite element to have elements of varying degrees. We can prove that the spaces constructed in both approaches are continuous as desired.

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