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TRƯỜNG ĐẠI HỌC DUY TÂN
DUYTAN UNIVERSITY

Mixture of Erlang and exponential approximation for ultimate ruin probability

Xấp xỉ hỗn tạp phân phối Erlang và phân phối mũ đối với xác suất phá sản của công ty bảo hiểm trong thời gian vô hạn

Tran Dong Xuan[a,b*]
Trần Đông Xuân[a,b*]

[a]*Institute of Fundamental and Applied Sciences, Duy Tan University, 6 Tran Nhat Duat, District 1, Ho Chi Minh City 700000, Vietnam*
[a]*Viện Nghiên cứu Khoa học Cơ bản và Ứng dụng, Đại học Duy Tân, 6 Trần Nhật Duật, Quận 1, TP. HCM*
[b]*Faculty of Natural Sciences, Duy Tan University, Da Nang City 550000, Viet Nam*
[b]*Khoa Khoa học Cơ bản, Đại học Duy Tân, Đà Nẵng 55000, Việt Nam*

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Abstract

In this paper, we investigate the mixture of Erlang and exponential approximation based on the first three moments and the matrix-exponential representation of mixed Erlang and exponential functions. Besides, a number of numerical examples are given to illustrate the quick convergence of the method in the cases of the first three moments of uniform distribution on $[0,1]$. Moreover to illustrate this method, a numerical example is given with different initial reserves u of insurance company.

Keyword: Mixed Erlang and exponential distributions; fitting moments; matrix-exponential representation; ultimate ruin probability; Cramér-Lundberg model.

Tóm tắt

Trong bài báo này, chúng tôi nghiên cứu xấp xỉ hỗn tạp phân phối Erlang và phân phối mũ nhờ vào ba mômen đầu tiên và biểu diễn mũ ma trận của hỗn tạp phân phối Erlang và phân phối mũ để tính xác suất phá sản của công ty bảo hiểm trong thời gian vô hạn. Bên cạnh đó, một ví dụ số đối với ba mômen của phân phối đều trên đoạn $[0,1]$ được đưa ra để minh họa tính hội tụ của phương pháp. Để minh họa cho phương pháp, một ví dụ số được trình bày với vốn ban đầu khác nhau của công ty bảo hiểm.

Từ khóa: Hỗn tạp phân phối Erlang và phân phối mũ; fitting mômen; biểu diễn mũ ma trận; xác suất phá sản trong thời gian vô hạn; mô hình Cramér-Lundberg.

[*]*Corresponding Author:* Tran Dong Xuan; Institute of Fundamental and Applied Sciences, Duy Tan University, 6 Tran Nhat Duat, District 1, Ho Chi Minh City 700000, Vietnam; Faculty of Natural Sciences, Duy Tan University, Da Nang City 550000, Viet Nam.
Email: trandongxuan@duytan.edu.vn

1. Introduction

In collective risk theory, the reserves process$\{U(t)\}_{t\geq 0}$of an insurance company is modeled as follows:

$$U(t) = u + ct - S(t), \quad t \geq 0 \quad (1)$$

where

- At time $t = 0$ the insurance company has initial capital $u \geq 0$;
- Premium income accrues linearly in time at rate $c = (1+\theta)\lambda m_1$, so that the total amount of premiums received is ct by the time t;
- θ is the relative security loading;
- $N(t)$ is the Poisson process with intensity λ which describes the number of claims in the interval $(0, t]$;
- $S(t) = \sum_{k=1}^{N(t)} Z_k$ is total amount claimed by the time t. The claims Z_k are independent identically distributed (iid) random variables independent of the $N(t)$, with cumulative distribution function (cdf) $F(t)$, probability density functions (pdf) $f(t)$ and mean $E(Z_k) = m_1$.

The ruin is said to occur if the insurer's surplus reaches a specified lower bound, e.g., minus the initial capital. One measure of risk is the probability of an event. It serves as a useful tool in long term planning for use of insurer's funds.

Let

$$\tau = \inf\{t \geq 0 : U(t) < 0\},$$

no comma and $X = -U(\tau)$ be the deficit at ruin or severity of ruin. We are interested in the ruin probability with initial reserves u, time span t and severity of ruin larger than x, i.e.,

$$\Psi(t, u, x) = \mathbb{P}_u\{\tau < t, X > x\}.$$

We will work a spectrally positive Lévy process $S(t)$, that is a Lévy process with only upward jumps that represent a cumulative number of claims up to time t. The Laplace exponential of the spectrally negative Lévy process $U(t)$ is denoted by $\kappa(\nu)$ satisfying

$$E[\exp(\nu U(t))] = \exp(\kappa(\nu)t)$$

to study the density of ruin probability with respect to $\psi(t, u, x) = \frac{\partial \Psi(t,u,x)}{\partial u}$ and it's more tractable Laplace transform in initial reserve u.

The approximated problem of ruin probabilities $\psi(u) = \Psi'(u)$ for Cramer-Lundberg model (1) using data on the claims is a classic of applied probability. It began in the early [1, 2].

Our goal is to approximate a given density function $f(t)$of the claims by mixed Erlang and exponential distributions

$$f(y) = \frac{\alpha_1 (\lambda_1 y)^{n-1} \exp(\lambda_1 y)}{(n-1)!}\lambda_1 + \alpha_2 \exp(\lambda_2 y), \quad \alpha_1 + \alpha_2 = 1 \quad (2)$$

which has three free parameters. This approximation is expressed in the matrix-exponential form as

$$f(y) = \boldsymbol{\alpha}\exp(\mathbf{T}y)\mathbf{t}.$$

Here, $\boldsymbol{\alpha}$ is the n initial state row vector, $\mathbf{T}$ is the $n \times n$ generator matrix of its underlying Markov chain and $\mathbf{t}$ is a n dimensional column vector. This representative method is used to calculate ultimate ruin probabilities.

The paper is organized as follows: Section 1 introduces the risk model of insurance company. Section 2 presents method of fitting the mixture of Erlang and exponential distributions by the first three moments of the claims in ruin process. Besides, examples are given to illustrate this approximated algorithm. The numerical results to illustrate the approximated method for the claims in the cases of uniform are given in Section 3. Finally, conclusions are drawn in Section 4.

2. Mixture of Erlang and exponential distribution

2.1. Matrix-exponential definition

A matrix-exponential (ME) distribution is defined as the distribution of the time until absorption in a finite-state Markov process with n transient states and an absorbing state, (n+1)-th state. The parameters of a ME distribution are its dimension n, the elements of an n-dimensional row vector $\boldsymbol{\alpha}$ and an $n \times n$ matrix $\mathbf{T}$.

Definition 1. A nonnegative random variable Y is distributed according to a ME distribution if its distribution has the form

$$F(y) = \begin{cases} \alpha_0, & y = 0 \\ 1 + \boldsymbol{\alpha}\exp(\mathbf{T}y)\mathbf{T}^{-1}\mathbf{t}, & y > 0' \end{cases} \quad (3)$$

where, for $n \geq 1$,

1. $\boldsymbol{\alpha}$ is a $1 \times n$ row vector,
2. $\mathbf{T}$ is a $n \times n$ matrix and
3. is a $n \times 1$ column vector.

It is immediately clear that $0 \leq \alpha_0 \leq 1$. A further stipulation on the parameters$\boldsymbol{\alpha}$,$\mathbf{T}$ and α_0 is that the distribution function (3) is right continuous for $y = 0$. That is

$$\lim_{y \to 0^+}(1 + \boldsymbol{\alpha}\exp(\mathbf{T}y)\mathbf{T}^{-1}\mathbf{t}) = \alpha_0.$$

The parameter α_0 is known as point mass at zero. The distribution is said to have matrix-exponential representation ($\boldsymbol{\alpha}$, $\mathbf{T}$, $\mathbf{t}$) of order n if its density function has the form

$$f(y) = \boldsymbol{\alpha}\exp(\mathbf{T}y)\mathbf{t}. \quad (4)$$

The Laplace-Stieltjes transform (LST) of (3) is given by

$$f^*(s) = E[exp(-sY)] = \int_0^\infty exp(-sy) f(y) dy = \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{t} + \alpha_0, \quad (5)$$

with $s \in \mathbb{C}$ such that $\mathcal{R}(s) > -\delta$ with δ is a positive number.

Differentiating (5) i times with respect to s and letting $s = 0$ gives, for $i = 1, 2, \ldots$, $E[T^i]$ is given by

$$m_i = (-1)^{i+1} i!\, \boldsymbol{\alpha}\mathbf{T}^{-(i+1)}\mathbf{t}, \quad (6)$$

Note. Exponential distribution with rate ϑ is matrix-exponential distribution with representation

$$\boldsymbol{\alpha} = 1; \quad \boldsymbol{T} = -\vartheta; \quad \boldsymbol{t} = \vartheta.$$

A Erlang distribution with order n is defined as the sum of n independent identical exponential distributed random variables.

2.2. Fitting the first three moments of mixed two Erlang distributions

In this part, we investigate fitting the first three moments' problem of a given density $f(y)$ of the claims by the mixture of Erlang and exponential distributions

$$f(y) = \alpha_1 f_1(y) + \alpha_2 f_2(y); \;\; \alpha_1 + \alpha_2 = 1$$

with λ_1 and n_1 is scale parameter and order of $f_1(y)$, and $f_2(y)$ is an exponential distribution with scale parameter λ_2. We also assume that $m_1, m_2, m_3$are the first three moments of density function $f(y)$.

The matching moment problem for mixed two Erlang distribution was extensively studied by Johnson and Taaffe and Thummler et al. [3, 4, 5], but determined the solution only for the case $n_1 = n_2 > n^*$, with

$$n^* = \left\lceil max\left\{\left(\frac{m_0 m_2}{{m_1}^2} - 1\right)^{-1}, \left(\frac{m_1 m_3}{{m_2}^2} - 1\right)^{-1}\right\}\right\rceil.$$

Within the scope of this article, we use numerical method to solve this problem in the case of $n_2 = 1 < n^* < n_1$. With this method, we can solve ruin probabilities of the insurance company faster than the method of Johnson and Taaffe and Thummler et al. because the order of matrix representation in our method is much smaller than the method of Johnson and Taaffe and Thummler et al... Now we illustrate our algorithm with a number of examples.

Example 1. Consider the uniform density with two parameters $a = 0$, $b = 1$ is given by

$$f(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

which has three moments

$$m_1 = \frac{1}{2}, m_2 = \frac{1}{3}, m_3 = \frac{1}{4}.$$

1) The mixture of two Erlang distribution of common order $n = n^* + 1 = 8$, we obtain unique feasible root as follows:

$$\lambda_1 = 202.67839,$$
$$\lambda_2 = 13.3216,$$
$$\alpha_1 = 0.17918.$$

So the density function of mixed two Erlang common order n is

$$f(t) = 1.01231 \times 10^{14} t^7 \exp(-202.67839\, t) + 1.61536 \times 10^5\, t^7 \exp(-13.32150\, t)$$

and its graph is plotted in Fig. 1a.

2) The mixture of Erlang $n = n^* + 1 = 8$ and exponential distributions, we obtain unique feasible root as follows:

$$\lambda_1 = 13.30939,$$
$$\lambda_2 = 22.32911,$$
$$\alpha_1 = 0.8183.$$

So the density function of mixed Erlang order n and exponential distributions is

$$f(t) = 1.59863 \times 10^5 y^7 \exp(-13.30939\, t) + 4.05722 \exp(-22.32911\, t)$$

and its graph is plotted in Fig. 1b.

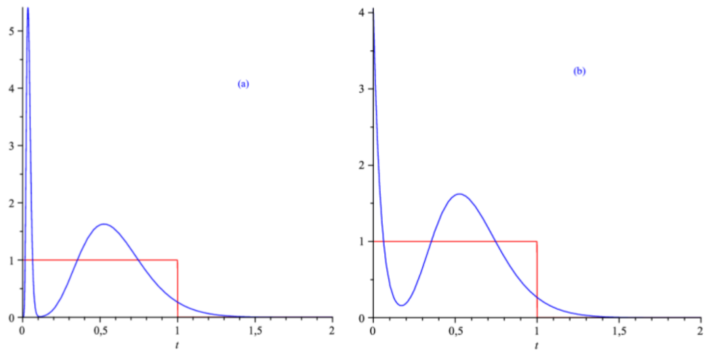

Fig. 1

3. Numerical result

In this section, we illustrate the algorithm numerically for ultimate ruin probabilities by the fitting of the first three moments method of the mixture of Erlang and exponential distributions presented with the different values of u. We assume that the claims arrival rate is $\lambda = 1$. We use $c = 1$ instead of the safe parameter θ, recall that $c = (1+\theta)\lambda m_1$. Applying the formula (17), page 135 of Xuan [6], the ultimate ruin probabilities for claim size distribution as Exp.1 are found as follows:

u	mixed two Erlang	Erlang + exponential
1	2.720766018e-01	3.722938376e-01
5	3.675303624e-05	5.028466393e-05
10	4.948874416e-09	6.776441472e-09
15	6.663764479e-13	9.132040463e-13
20	8.972900450e-17	1.230648331e-16
25	1.208220098e-20	1.658441292e-20
30	1.626894031e-24	2.234941923e-24
35	2.190647357e-28	3.011843334e-28
40	2.949753208e-32	4.058808088e-32
45	3.971905327e-36	5.469714429e-36

Tab.1: Ruin probabilities with the claims size distribution U[0,1]

4. Conclusion

In this article, we present fitting moments method to obtain the mixture of Erlang and exponential distributions. This method only uses the first three moments of the claims distribution. This is a generalization of Johnson, M.A. and Taaffe, M.R. (1989, 1990) who determined the solution for the case the mixture of two Erlang distributions with common order $(n_1 = n_2)$.

In the case, claim size is uniform distribution on $[0,1]$, the result of our method is better than the result of Padé approximation [7]. Because the result of Padé method is negative at a number of points, but our method present in this article is not negative. Non-negativity of the estimated function is checked by CheckMEPositiveDensity in Butools, see Bodrog, L. et al. Butools: Program packages for computations with PH, ME distributions and MAP, RAP processes (October 2011), http://webspn.hit.bme.hu/~butools.

Besides, with the method in this article, we can calculate ultimate ruin probabilities with the first three moments of the claim sizes while to calculate infinite-time ruin probabilities by the previous method as [8-13] and so on, we must know distribution of the claim sizes.

Moreover, the numerical result of ultimate ruin probabilities can be easily calculated and the explicit solutions of infinite-time ruin probabilities can also be found in the cases, the claim sizes have exponential, sum of exponential, Erlang, mixed Erlang distributions, etc by the method in this article.

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