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Asymptotic behavior of solutions of a periodically nonlinear elasticcity problem

Dáng điệu tiệm cận nghiệm của bài toán đàn hồi phi tuyến tuần hoàn

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Abstract

We study asymptotic behavior of solutions of a periodically nonlinear elasticity problem in one-dimensional and strain-limiting settings.

Keywords: Asymptotic behavior, homogenization, periodic, nonlinear elasticity, strain-limiting.

Tóm tắt

Chúng tôi nghiên cứu dáng điệu tiệm cận nghiệm của bài toán đàn hồi phi tuyến tuần hoàn trong thiết lập một chiều và giới hạn biến dạng.

Từ khóa: Dáng điệu tiệm cận, đồng nhất hóa, tuần hoàn, độ đàn hồi phi tuyến, giới hạn biến dạng.

1. Introduction

As a model reduction approach for tackling multiscale problems, homogenization means upscaling the material properties to capture macroscopic behaviors. Toward homogenization investigation of our considering nonlinear elasticity models, we focus on a periodic strain-limiting problem. (The strain-limiting parameter in this paper is a function depending on the position variable, which is different from the constant in [1, 2].) In particular, we study asymptotic behavior of solutions of a periodically nonlinear elasticity problem in one-dimensional and strain-limiting settings.

2. Formulation of the problem

2.1. Classical formulation

We consider, as in Figure 1 in the x-direction, the spatially periodic 1D composite rod consisting of alternating layers of nonlinear elastic materials $\Omega^{(1)}$ and $\Omega^{(2)}$. The microscopic size l corresponds to the length of a periodically repeated base cell. The macroscopic size of the entire

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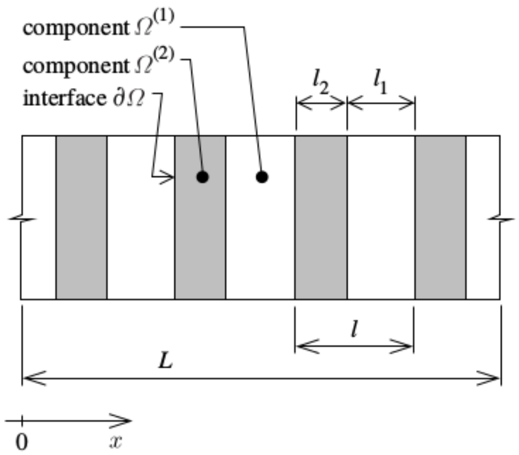

Hình 1. Layered composite structure (from [3]).

sampling $\Omega \subset \mathbb{R}$ of the rod is denoted by L. Without loss of generality, we choose $l = \epsilon$ (the period of the structure) and take $L = 1$ so that

$$\epsilon = \frac{l}{L} = \frac{\epsilon}{1} = \frac{k\epsilon}{k1}. \tag{1}$$

Here, $\dfrac{x}{\epsilon}$ represents the local position.

We assume that the rod is at a static state after the action of body forces (along the rod) $f : \Omega \to \mathbb{R}$ and traction forces $G : \partial\Omega_T \to \mathbb{R}$. The boundary of the set Ω is denoted by $\partial\Omega$. It is Lipschitz continuous and consists of two parts $\partial\Omega_T$ and $\partial\Omega_D$. The displacement $u : \Omega \to \mathbb{R}$ is provided on $\partial\Omega_D$. We are considering the strain-limiting model of the form (as in [1])

$$E = \frac{\sigma}{1 + \beta(x)|\sigma|}. \tag{2}$$

Equivalently,

$$\sigma = \frac{E}{1 - \beta(x)|E|}. \tag{3}$$

In Eqs. (2) and (3), $\beta(x)$ will be introduced in the next paragraph, σ stands for the Cauchy stress $\sigma : \Omega \to \mathbb{R}$; and E represents the classical linearized strain tensor

$$E := \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}}). \tag{4}$$

In one-dimensional setting, it is

$$E := u', \tag{5}$$

that is, the spatial derivative of u. Hence, by (3),

$$\sigma = \frac{u'}{1 - \beta(x)|u'|}. \tag{6}$$

The strain-limiting parameter function is represented by $\beta(x)$, which depends on the position variable x, and it is constant over each layer, with $\beta_\epsilon(x) = \beta(\epsilon^{-1}x)$. We obtain from (2) that

$$|E| = \frac{|\sigma|}{1 + \beta(x)|\sigma|} < \frac{1}{\beta(x)}. \tag{7}$$

This implies that $\dfrac{1}{\beta(x)}$ is the upper-bound on $|E|$ and taking sufficiently large $\beta(x)$ gives the limiting-strain small upper-bound, as desired. However, we stay away from too large $\beta(x)$. If $\beta(x) \to \infty$ then $|E| < \dfrac{1}{\beta(x)} \to 0$, a contradiction. Moreover, we assume that $\beta(x)$ is smooth and have compact range $0 < m \le \beta(x) \le M$. Also, it is assumed that

$$\beta(x) = \begin{cases} \beta_1 & \text{if } jl < x < (j+\alpha)l \text{ for some } j \in \mathbb{N}, \\ \beta_2 & \text{otherwise}. \end{cases} \tag{8}$$

Here, β_1 and β_2 are taken so that the strong ellipticity condition [1] is satisfied. Practically, the requirement of strong point-wise ellipticity in each layer is not necessary. The reason is that all the important instability phenomena occur rather below the stress levels corresponding to the loss of ellipticity of the weakest layer (see [4, 5]).

2.2. Function spaces

Our considered space is $V := H_0^1(\Omega)$. Nevertheless, the methods here can be applied to more general space $H_0^p(\Omega)$, where $2 \le p < \infty$. The space $W_0^{1,2}(\Omega)$ is of interest because we can describe displacements that vanish on the boundary $\partial\Omega$ of Ω.

We denote by $H^{-1}(\Omega)$ the dual space, which is the space of continuous linear functionals on $H_0^1(\Omega)$, and the value of a functional $b \in H^{-1}(\Omega)$ at a point $v \in H_0^1(\Omega)$ is represented by $\langle b, v\rangle$. The Sobolev norm $\|\cdot\|_{H_0^1(\Omega)}$ is of the form

$$\|v\|_{H_0^1(\Omega)} = \|v\|_{H^1(\Omega)} := \left(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

The dual norm to $\|\cdot\|_{H_0^1(\Omega)}$ is $\|\cdot\|_{H^{-1}(\Omega)}$.

Let Ω be a bounded, connected, open, Lipschitz domain of $\mathbb{R}$,

$$f \in H_*^1(\Omega) = \left\{ g \in H^1(\Omega) \middle| \int_\Omega g\,dx = 0 \right\}.$$

We consider the following problem: Find $u \in H^1(\Omega)$ and $\sigma \in L^1(\Omega)$ ([6]) such that

$$\begin{aligned} -\mathrm{div}(\sigma) &= f \quad \text{in } \Omega, \\ \sigma &= \frac{u'}{1-\beta(x)|u'|} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega_D, \\ \sigma &= G \quad \text{on } \partial\Omega_T. \end{aligned} \tag{9}$$

The considered model (2) is compatible with the laws of thermodynamics [7, 8], that is, the class of materials are elastic and non-dissipative.

For the later use, we consider $u_\epsilon(x) \in W_0^{1,2}(\Omega)$. Assume that $u_\epsilon(x) = u\left(\frac{x}{\epsilon}\right)$ is a periodic function in x with period ϵ. Equivalently, $u(y) = u\left(\frac{x}{\epsilon}\right)$ is a periodic function in y with period 1. This implies that for any integer k,

$$u_\epsilon(x) = u_\epsilon(x+\epsilon) = u_\epsilon(x+k\epsilon),$$

correspondingly,

$$u\left(\frac{x}{\epsilon}\right) = u\left(\frac{x}{\epsilon}+1\right) = u\left(\frac{x}{\epsilon}+k1\right) = u(y+k).$$

This observation supports the expressions of ϵ in (1). (Note that the spatial periodicity of the composite produces the same periodicity for u.)

For simplicity, we assume perfect bonding conditions at the interface $\partial\Omega$ between the layers, that is, the displacement and traction are continuous across each interface for all possible deformations:

$$\begin{aligned} (u_\epsilon)_{(1)} &= (u_\epsilon)_{(2)} \quad \text{on } \partial\Omega, \\ (\sigma_\epsilon)_{(1)} &= (\sigma_\epsilon)_{(2)} \quad \text{on } \partial\Omega_T. \end{aligned} \tag{10}$$

We assume $\partial\Omega_T = \emptyset$. In homogenization theory, using (9), we rewrite the considered formulation in the form of displacement problem: Find $u \in H^1(\Omega)$ such that

$$-\mathrm{div}\left(\frac{u'_\epsilon}{1-\beta_\epsilon(x)\left|u'_\epsilon\right|}\right) = f \quad \text{in } \Omega, \tag{11}$$

$$u_\epsilon = 0 \quad (u_\epsilon)_{(1)} = (u_\epsilon)_{(2)} \quad \text{on } \partial\Omega. \tag{12}$$

Let

$$a_\epsilon(x, u'_\epsilon) = \frac{u'_\epsilon}{1-\beta_\epsilon(x)\left|u'_\epsilon\right|}, \tag{13}$$

in which $u_\epsilon(x) \in W_0^{1,2}(\Omega)$.

3. Existence and uniqueness

In [9], the existence and uniqueness of solution to (11)-(12) is proved and thanks to the following Lemma ([9, 10, 11]).

Lemma 3.1. *Let*

$$\mathcal{Z} := \left\{ \zeta \in L^\infty(\Omega) \,\middle|\, 0 \le |\zeta| < \frac{1}{M} \right\}. \tag{14}$$

For any $\xi \in \mathcal{Z}$, consider the mapping

$$\xi \in \mathcal{Z} \mapsto F(\xi) := \frac{\xi}{1-\beta_\epsilon(x)|\xi|} \in \mathbb{R}.$$

Then, for each $\xi_1, \xi_2 \in \mathcal{Z}$, we have

$$|F(\xi_1) - F(\xi_2)| \le \frac{|\xi_1 - \xi_2|}{(1-\beta_\epsilon(x)(|\xi_1|+|\xi_2|))^2}, \tag{15}$$

$$(F(\xi_1) - F(\xi_2))(\xi_1 - \xi_2) \ge |\xi_1 - \xi_2|^2. \tag{16}$$

In our case of 1D, the solution u can be obtained directly from (11)-(12).

4. Asymptotic behavior of solutions

Now, we want to investigate the asymptotic behavior of the solutions u_ϵ of the following problem (in periodic case)

$$\begin{cases} -\operatorname{div} a\left(\frac{x}{\epsilon}, u'_\epsilon\right) = f \text{ on } \Omega, \\ u_\epsilon \in H^1_0(\Omega), \end{cases} \tag{17}$$

as $\epsilon \to 0$. We will prove that u_ϵ converges weakly in $H^1_0(\Omega)$ to the solution u_* of the problem

$$\begin{cases} -\operatorname{div} \hat{a}\left(u'_*\right) = f \quad \text{ on } \Omega, \\ u_* \in H^1_0(\Omega), \end{cases} \tag{18}$$

whose representation can be obtained from a.

The weak formulation of (17) is as follows:

$$\begin{cases} \int_\Omega \left(a\left(\frac{x}{\epsilon}, u'_\epsilon\right)\right)\phi'\,dx = \int_\Omega f\phi\,dx, \forall\phi \in V, \\ u_\epsilon \in H^1_0(\Omega). \end{cases} \tag{19}$$

Let Y be the unit period in $\mathbb{R}$. We denote by $W^{1,2}_{\text{per}}(Y)$ the set of all mean value zero functions in the Sobolev space $W^{1,2}(Y)$. The homogenization results for periodic case are stated and proved below, thanks to [12, 13, 14].

Theorem 4.1 ([13]). *Let u_ϵ be the solutions of (19), where a is 1-periodic, piecewise continuous in the first variable, and satisfies the boundedness $a(x,0) = 0$ as well as continuity condition (16) and monotonicity condition (15) on the second variable. Then,*

$$u_\epsilon \rightharpoonup u_0 \text{ in } H^1_0(\Omega),$$
$$a\left(\frac{x}{\epsilon}, u'_\epsilon\right) \rightharpoonup \hat{a}(u'_0) \text{ in } L^2(\Omega),$$

as $\epsilon \to 0$, where u_0 is the unique solution of

$$\begin{cases} \int_\Omega \left(\hat{a}\left(u'_0\right)\right)\phi'\,dx = \int_\Omega f\phi\,dx \;\; \forall\phi \in H^1_0(\Omega), \\ u_0 \in H^1_0(\Omega). \end{cases} \tag{20}$$

The operator $\hat{a}$ is defined as

$$\hat{a}(\xi) = \int_Y a\left(y, \xi + D_y v^\xi\right) dy, \tag{21}$$

where v^ξ is the unique solution of the cell problem

$$\begin{cases} \int_Y \left(a\left(y, \xi + D_y v^\xi\right)\right)\phi\,dy = 0 \;\forall\phi \in W^{1,2}_{per}(Y), \\ v^\xi \in W^{1,2}_{per}(Y). \end{cases} \tag{22}$$

Proof. First, we note that u_ϵ and $a\left(\frac{x}{\epsilon}, u'_\epsilon\right)$ are bounded in $H^1(\Omega)$ and $L^2(\Omega)$, respectively. Indeed, let $\phi = u_\epsilon$ in (19), then it follows from the coercivity of a and (19) that

$$\begin{aligned} \|u'_\epsilon\|^2_{L^2(\Omega)} &= \int_\Omega |u'_\epsilon|^2\,dx \\ &\le \int_\Omega a\left(\frac{x}{\epsilon}, u'_\epsilon\right) u'_\epsilon\,dx \\ &\le \|f\|_{H^{-1}(\Omega)}\|u_\epsilon\|_{H^1(\Omega)} \\ &\le c\|u_\epsilon\|_{H^1(\Omega)}. \end{aligned} \tag{23}$$

The Poincaré inequality

$$\|u_\epsilon\|_{L^2(\Omega)} \le \left\|u'_\epsilon\right\|_{L^2(\Omega)}$$

leads to

$$\begin{aligned} \left\|u'_\epsilon\right\|_{L^2(\Omega)} \le \|u_\epsilon\|_{H^1(\Omega)} &= \left(\|u_\epsilon\|^2_{L^2(\Omega)} + \left\|u'_\epsilon\right\|^2_{L^2(\Omega)}\right)^{1/2} \\ &\le \sqrt{2}\left\|u'_\epsilon\right\|_{L^2(\Omega)}. \end{aligned}$$

This means that the norms $\left\|u'_\epsilon\right\|_{L^2(\Omega)}$ and $\|u_\epsilon\|_{H^1(\Omega)}$ on $H^1_0(\Omega)$ are equivalent. Thus,

$$\frac{1}{2}\|u_\epsilon\|^2_{H^1(\Omega)} \le \left\|u'_\epsilon\right\|^2_{L^2(\Omega)}.$$

Taking (23) into account, we obtain

$$\frac{1}{2}\|u_\epsilon\|^2_{H^1(\Omega)} \le \left\|u'_\epsilon\right\|^2_{L^2(\Omega)} \le c\|u_\epsilon\|_{H^1(\Omega)},$$

which implies

$$\|u_\epsilon\|_{H^1(\Omega)} \le 2c.$$

The desired boundedness of the sequence u_ϵ in $H^1(\Omega)$ then follows. Thus, there exists a subsequence, still denoted by u_ϵ such that

$$u_\epsilon \rightharpoonup u_* \text{ in } H^1_0(\Omega).$$

It follows that $a\left(\frac{x}{\epsilon}, u'_\epsilon\right)$ is bounded in $L^2(\Omega)$. Indeed, by the boundedness $a(x,0)=0$ and the growth condition of a and (23), we obtain

$$\left\|a\left(\frac{x}{\epsilon}, u'_\epsilon\right)\right\|^2_{L^2(\Omega)} = \int_\Omega \left|a\left(\frac{x}{\epsilon}, u'_\epsilon\right)\right|^2 dx \tag{24}$$
$$\leq c\int_\Omega |u'_\epsilon|^2\, dx$$
$$\leq c\|u_\epsilon\|^2_{H^1(\Omega)}$$
$$\leq C,$$

where the constant C is independent of ϵ. This means that there is a subsequence, still denoted by $a\left(\frac{x}{\epsilon}, u'_\epsilon\right)$ such that

$$a\left(\frac{x}{\epsilon}, u'_\epsilon\right) \rightharpoonup \eta_*(x) \text{ in } L^2(\Omega),$$

One can show (using the ideas from the proof of Theorem 11.2 in [15]) that

$$\eta_*(x)(=\hat{a}(u_*)) = \tilde{a}(x, u'_*),\ \text{a.e. in } \Omega$$

for some $\tilde{a} \in Mon(1,\alpha;\Omega)$ (notation in [15]), and the following equation is satisfied (see [14]):

$$-\operatorname{div}\eta_*(x) = f \text{ on } \Omega.$$

that is,

$$-\operatorname{div}\tilde{a}(x, u'_*) = f \text{ on } \Omega,$$

with the unique solution u_*.

In our special case $p=2$ for (19), the existence and uniqueness of weak solution has been verified in [10]. Also, from (19), we have that

$$\begin{cases} \int_\Omega \left(a\left(\frac{x}{\epsilon}, u'_\epsilon\right)\right)\phi'\, dx = \int_\Omega f\phi\, dx,\ \forall \phi \in V, \\ u_\epsilon \in H^1_0(\Omega). \end{cases} \tag{25}$$

Passing to limit when $\epsilon \to 0$, we obtain

$$\int_\Omega \eta_*\phi'\, dx = \int_\Omega f\phi\, dx, \quad \forall \phi \in H^1_0(\Omega).$$

This means especially that (see [13])

$$\int_\Omega \eta_*\phi'\, dx = \int_\Omega f\phi\, dx, \quad \forall \phi \in C_0^\infty(\Omega).$$

If we can show that

$$\eta_* = \hat{a}\left(u'_*\right), \quad \text{for a.e. } x\in\Omega, \tag{26}$$

then it follows by the uniqueness of the solution of the homogenized problem (20) that $u_* = u_0$.

To this end, we fix ξ and let u^ξ_ϵ be defined as the unique solution of the auxiliary problem

$$\begin{cases} \int_Y \left(a\left(y, \xi + Du^\xi_\epsilon\right)\right)\phi'\, dx = 0,\ \forall \phi \in W^{1,2}_{\text{per}}(Y), \\ u^\xi_\epsilon \in W^{1,2}_{\text{per}}(Y), \end{cases} \tag{27}$$

such that

$$\hat{a}(\xi) = \int_Y a\left(y, \xi + Du^\xi_\epsilon\right) dy,$$

(recall that $\hat{a}$ was defined in (21)).

Now, we define

$$w^\xi_\epsilon(x) = \xi x + \epsilon u^\xi_\epsilon\left(\frac{x}{\epsilon}\right).$$

Then,

$$\begin{cases} w^\xi_\epsilon \rightharpoonup \xi x \quad \text{in } H^1(\Omega), \\ D_x w^\xi_\epsilon \rightharpoonup \xi \quad \text{in } L^2(\Omega), \\ a\left(\frac{x}{\epsilon}, Dw^\xi_\epsilon\right) \rightharpoonup \hat{a}(\xi) \quad \text{in } L^2(\Omega), \\ -\operatorname{div}_x a\left(\frac{x}{\epsilon}, Dw^\xi_\epsilon\right) = 0 \text{ on } \Omega. \end{cases}$$

Based on the monotonicity of a, we have

$$\int_\Omega \left(a\left(\frac{x}{\epsilon}, Du_\epsilon\right) - a\left(\frac{x}{\epsilon}, Dw^\xi_\epsilon\right)\right)(Du_\epsilon - Dw^\xi_\epsilon)\phi \geq 0,$$

for any nonnegative $\phi \in C_0^\infty(\Omega)$.

The compensated compactness (Div-Curl Lemma) and periodicity then implies that

$$\int_\Omega (\eta_*(x) - \hat{a}(\xi))(Du_* - \xi)\phi\, dx \geq 0,$$

for any nonnegative $\phi \in C_0^\infty(\Omega)$. Hence, for a fix $\xi \in \mathbb{R}$ as in our setting, we have that

$$(\eta_*(x) - \hat{a}(\xi))(Du_*(x) - \xi) \geq 0 \quad \text{for a.e. } x\in\Omega. \tag{28}$$

In particular, if (ξ_m) is a countable dense subset in $\mathbb{R}$, then (28) implies that

$$(\eta_*(x)-\hat{a}(\xi_m))(Du_*(x)-\xi_m) \geq 0 \quad \text{for a.e. } x \in \Omega. \tag{29}$$

By the continuity of $\hat{a}$ (readily), it follows that

$$(\eta_*(x)-\hat{a}(\xi))(Du_*(x)-\xi) \geq 0 \quad \text{for a.e. } x \in \Omega,$$

and for every $\xi \in \mathbb{R}$. Since $\hat{a}$ is monotone and continuous, we have that $\hat{a}$ is maximal monotone. This means $\eta_*(x) = \hat{a}(Du_*)$, and we obtain the desired result. □

5. Conclusions

In this paper, we investigate asymptotic behavior of solutions for a periodically nonlinear elasticity problem in one-dimensional and strain-limiting settings. By analysis, we obtained the limit of the solutions. An open question is extending this study to higher dimensions and more general settings.

References

[1] Tina Mai and Jay R. Walton. On strong ellipticity for implicit and strain-limiting theories of elasticity. *Mathematics and Mechanics of Solids*, 20(II):121–139, 2015. DOI: 10.1177/1081286514544254.

[2] Tina Mai and Jay R. Walton. On monotonicity for strain-limiting theories of elasticity. *Journal of Elasticity*, 120(I):39–65, 2015. DOI: 10.1007/s10659-014-9503-4.

[3] Igor V Andrianov, Vladimir I Bolshakov, Vladyslav V Danishevs'kyy, and Dieter Weichert. Higher order asymptotic homogenization and wave propagation in periodic composite materials. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 464(2093):1181–1201, 2008.

[4] Giuseppe Geymonat, Stefan Müller, and Nicolas Triantafyllidis. Homogenization of nonlinearly elastic materials, microscopic bifurcation and macroscopic loss of rank-one convexity. *Archive for Rational Mechanics and Analysis*, 122(3):231–290, 1993. DOI: 10.1007/BF00380256.

[5] N. Triantafyllidis and B.N. Maker. On the comparison between microscopic and macroscopic instability mechanisms in a class of fiber-reinforced composites. *J. Appl. Mech*, 52(4):794 – 800, 1985.

[6] Lisa Beck, Miroslav Bulíček, Josef Málek, and Endre Süli. On the existence of integrable solutions to nonlinear elliptic systems and variational problems with linear growth. *Archive for Rational Mechanics and Analysis*, 225(2):717–769, Aug 2017.

[7] K. R. Rajagopal and A. R. Srinivasa. On the response of non-dissipative solids. *Proceedings of the Royal Society of London, Mathematical, Physical and Engineering Sciences*, 463(2078):357–367, 2007.

[8] K.R Rajagopal and A.R Srinivasa. On a class of non-dissipative materials that are not hyperelastic. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 465(2102):493–500, 2009.

[9] Shubin Fu, Eric Chung, and Tina Mai. Generalized multiscale finite element method for a strain-limiting nonlinear elasticity model. *Journal of Computational and Applied Mathematics*, 359:153 – 165, 2019.

[10] M. Bulíček, J. Málek, and E. Süli. Analysis and approximation of a strain-limiting nonlinear elastic model. *Mathematics and Mechanics of Solids*, 20(I):92–118, 2015. DOI: 10.1177/1081286514543601.

[11] Miroslav Bulíček, Josef Málek, K. R. Rajagopal, and Endre Süli. On elastic solids with limiting small strain: modelling and analysis. *EMS Surveys in Mathematical Sciences*, 1(2):283–332, 2014.

[12] N. Fusco and G. Moscariello. On the homogenization of quasilinear divergence structure operators. *Ann. Math. Pura Appl.*, 146(4):1–13, 1987.

[13] J. L. Lions, D. Lukkassen, L. E. Persson, and P. Wall. Reiterated homogenization of nonlinear monotone operators. *Chinese Annals of Mathematics*, 22(01):1–12, 2001.

[14] Alexander Pankov. *G-convergence and homogenization of nonlinear partial differential operators*, volume 422 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1997.

[15] Luc Tartar. *The general theory of homogenization*, volume 7 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin; UMI, Bologna, 2009. A personalized introduction.