On finite-time attractivity for semilinear generalized rayleigh-stokes equation involving delays

Do Lan[1*], Nguyen Thi Ly[1], Pham Nam Giang[1], Vu Nam Phong[1]

Abstract: In this article, a functional semilinear Rayleigh-Stokes equation involving a fractional derivative with delay is investigated. The paper employs the time-fractional derivative which is Riemann-Liouville type. By using the fixed point argument and some useful estimates, the writers derive some global existence results. Also, the finite-time attractivity of all solutions are proved thanks to a Halanay- type inequality.

Keyword: Rayleigh-Stokes equation, finite-delay, finite-time stability.

1. Introduction

Let $\Omega \subset \mathrm{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, the following generalized Rayleigh-Stokes problem with respect to a time-fractional derivative and a nonlinear source term is considered

$$\begin{cases} \partial_t x - (1+\gamma\partial_t^\alpha)\Delta x = f(t, x_\rho) \text{ in } \Omega, t>0, & (1.1) \\ x = 0 \text{ on } \partial\Omega, t \ge 0, & (1.2) \\ x(u,s) = \xi(u,s), u \in \Omega, s \in [-h,0], & (1.3) \end{cases}$$

where $\gamma > 0$, $\alpha \in (0,1)$, $\partial_t = \dfrac{\partial}{\partial t}$, the notation ∂_t^α is the Riemann-Liouville derivative of order α defined by $\partial_t^\alpha v(x,t) = \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\int_0^t (t-s)^\alpha v(x,s)ds$, where $\Gamma(t)$ is the Gamma function. In our problem, x_ρ is defined by $x_\rho(u,t) = x(u, t-\rho(t))$ with ρ being a continuous function on R^+ such that $-h \le t - \rho(t) \le t$ and $\lim_{t\to\infty}(t-\rho(t)) = \infty$, f: $\mathrm{R}^+ \times L^2(\Omega) \to L^2(\Omega)$ is a nonlinear map and $\xi \in C([-h,0]; L^2(\Omega))$ is given.

Indeed, many aspects of science have witnessed and recognized the application of fractional calculus. Therefore, a range of papers on partial differential equations with fractional derivatives have been published by dedicated mathematicians so far (see (Bajlekova 2001, Kilbas et al 2006, Lan et al 2021-2022)). The generalized Rayleigh-Stokes equation employed to illustrate the non-Newtonian behaviour of fluids occupied in cylinders (see (Shen et al 2006)) and other applications of this equation can be found in (Fetecau et al 2009, Shen et al 2006). As a matter of fact, mathematicians have developed different numerical methods to solve Rayleigh-Stokes problem in the linear case, see e.g. (Bajeckova et al 2015, Chen et al 2013). Moreover, the analytic representation for solution of this problem can be found in (Fetecau et al 2009, Shen et al 2006). Recently, some inverse problems involving (1.1) has been studied in (Ngoc et al 2021, Tuan et al 2019) in the case without delay.

To the knowledge of the authors, so far, there is only one result on the stability of the generalized Rayleigh-Stokes when there is no

[1]Department of Mathematics, Thuyloi University, Hanoi, Vietnam
* Corresponding author
Received 4th Apr. 2022
Accepted 21st May 2022
Available online 31st Dec. 2022

delay (Lan 2022). In this work, the authors proves some results of asymptotic stability and the existence of decay solutions.

Up to now, the problem with finite delay has not been studied. Compared with problems without delay, the problem with delay will be much more complicated, because the inequalities of the Halanay form will have to be applied in the stability calculation. That is the main reason why we chose to study this problem.

In this paper, we examine the following concept of finite time stability when they study the stability in finite time. This definition helps the authors to have a clear view of the attractiveness after a period of time T.

Definition 1.1. *Let x and x' be the solution of (1.1) with respect to the initial datum ψ and ψ'. Then x is said to be attractive on $[0,T]$ if $\|x(t)-x'(t)\| < \|\psi-\psi'\|_\infty$, for all $t\in[0,T]$, and $\|\cdot\|_\infty$ denote the sup norm in $C([-h,0];L^2(\Omega))$.*

The main goal of this paper is to prove the finite-time stability of every solution to our problem provided that the nonlinear Lipschitz part is as shown in the following theorem

Theorem 1.1. *Let $f:[0,T]\times L^2(\Omega)\to L^2(\Omega)$ be a continuous mapping such that $p\|v-u\| \ge \|f(t,v)-f(t,u)\|$ for all $u,v\in L^2(\Omega)$ and the constant $p<r\lambda_1$, where r is the solution in $(0,1)$ of cubic equation $y^3-3y+1=0$. If x is the solution with respect to the initial datum ξ, then x is attractive on $[0,T]$ for all $\xi\in C([-h,0];L^2(\Omega))$.*

The paper includes three main sections. Section 2 supplies some basic settings, including some typical properties of relaxation function and the compactness of the Cauchy operator. In Section 3, some existence results of the mild solution together with some cases of the nonlinear source term are investigated. In Section 4, we consider the finite-time attractivity of all solutions to our problem.

2. Preliminaries

In this section, we first present the relaxation problem

$$\omega'(t)+\mu(1+\gamma\partial_t^\alpha)\omega(t)=0, t>0, \tag{2.1}$$

$$\omega(0)=1, \tag{2.2}$$

where $\mu,\gamma>0$ and ω is a scalar function.

Let $\omega(\cdot,\mu)$ be the notation of the solution of (2.1) and (2.2). Now, we show some properties of $\omega(\cdot,\mu)$ in the following proposition. We can see the proof of this proposition in (Lan 2022).

Proposition 2.1. *The solution $\omega(t)$ of* (2.1) and (2.2) *have the following properties:*

1) $0<\omega(t)\le 1$ for all $t\ge 0$ and ω is nonincreasing.

2) The function ω is completely monotone for $t\ge0$, i.e. $(-1)^n\omega^{(n)}(t)\ge 0$ for $t\ge0$ and $n\in N$.

3) $\omega(t)\le \mu^{-1}\min\{t^{-1},t^{\alpha-1}\}$, for all $t>0$.

4) $\int_0^t \omega(t-s)ds=\int_0^t\omega(s)ds\le \frac{1-\omega(t)}{\mu}$, for any $t>0$.

5) For fixed $t\ge0$ and $\gamma>0$, the function $\mu\mapsto\omega(t,\mu)$ is nonincreasing on $[0,\infty)$.

Second, we consider the inhomogeneous problem

$$y'(t)+\mu(1+\gamma\partial_t^\alpha)y(t)=f(t), t>0, \tag{2.3}$$

$$y(0)=y_0, \tag{2.4}$$

where $\mu,\gamma>0$ and f is a continuous function.

Here, we define $F*G$ as a Laplace convolution of F and G as

$$(F*G)(t)=\int_0^t F(t-s)G(s)ds, F,G\in L^1_{loc}(\mathrm{R}^+)$$

The next proposition shows the solution of the inhomogeneous problem and the proof can be found in (Lan 2022).

Proposition 2.2. *The solution of* (2.3) and (2.4) *is given by*

$$y(t)=\omega(t,\mu)y_0+\omega(\cdot,\mu)*f(t), t\ge 0,$$

where ω is the solution of (2.1) *and* (2.2).

Let $\{\varphi_n\}_{n=1}^\infty$ be a orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Laplacian $-\Delta$ subject to homogeneous Dirichlet boundary condition, that is $-\Delta\varphi_n=\lambda_n\varphi_n$ in Ω

and $\varphi_n = 0$ on $\partial\Omega$, where we can assume that $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence, $\lambda_n > 0$ and $\lambda_n \to +\infty$ as $n \to \infty$. In what follows, we consider the linear problem

$$\partial_t x - (1+\gamma\partial_t^{\alpha})\Delta x = g \text{ in } \Omega, t > 0, \quad (2.5)$$
$$x = 0 \text{ on } \partial\Omega, t \geq 0, \quad (2.6)$$
$$x(\cdot,0) = \xi \text{ in } \Omega, \quad (2.7)$$

where $g \in L^1_{loc}(\mathrm{R}^+; L^2(\Omega))$ and $\xi \in L^2(\Omega)$.

By letting $x(u,t) = \sum_{n=1}^{\infty} x_n(t)\varphi_n(u), g(u,t) = \sum_{n=1}^{\infty} g_n(t)\varphi_n(u), \xi(u) = \sum_{n=1}^{\infty} \xi_n\varphi_n(u)$, we get $x_{n'}(t) + \lambda_n(1+\gamma\partial_t^{\alpha})x_n(t) = g_n(t)$, $x_n(0) = \xi_n$.

Using Proposition 2.2, we obtain

$$x_n(t) = \omega(t,\lambda_n)\xi_n + \int_0^t \omega(t-s,\lambda_n) g_n(s) ds.$$

We define resolvent operator $S(t): L^2(\Omega) \to L^2(\Omega)$, by $S(t)\xi = \sum_{n=1}^{\infty} \omega(t,\lambda_n)\xi_n\varphi_n$. Based on this, the solution of the linear problem can be represented by

$$x(t) = S(t)\xi + \int_0^t S(t-s) g(s) ds. \quad (2.8)$$

Here and hereafter, we use the notation $\|\cdot\|$ for the standard norm in $L^2(\Omega)$ and the notation $\|\cdot\|_{\mathrm{L}}$ for the operator norm of bounded linear operators on $L^2(\Omega)$. We use the notation $\|\cdot\|_{\infty}$ for the sup norm in $C([-h,0]; L^2(\Omega))$, $C([-h,T]; L^2(\Omega))$ and $C([0,T]; L^2(\Omega))$, i.e. with $g \in C([0,T]; L^2(\Omega))$, we have $\|g\|_{\infty} = \sup_{t\in[0,T]} \|g(t)\|$. Now, we introduce some properties of the resolvent operator $S(\cdot)$ in the following lemma.

Lemma 2.3. *For any* $x \in L^2(\Omega)$, $T > 0$, *we have:*

1) $S(\cdot)x \in C((0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$.

2) $\|S(t)x\| \leq \omega(t,\lambda_1)\|x\|$, *for all* $t \geq 0$. *In particular,* $\|S(t)\| \leq 1$ *for all* $t \geq 0$.

3) $S(\cdot)x \in C^{(m)}((0,T]; L^2(\Omega))$ *for all* $m \in N$, *and* $\|S^{(m)}(t)x\| \leq \kappa t^{-m}\|x\|$, *where* κ *is a positive constant.*

4) $\|\Delta S^{(m)}(t)x\| \leq \kappa t^{-m-1+\alpha}\|x\|$ *for all* $t > 0$ *and* $m \in N$.

We define the *Cauchy operator* $Q: C([0,T]; L^2(\Omega)) \to C([0,T]; L^2(\Omega))$ by

$$Q(f)(t) = \int_0^t S(t-s) f(s) ds. \quad (2.9)$$

The compactness of Q plays an important role in proving the existence results in the next section. We introduce the lemma below, that the proof of this lemma similar in (Lan 2022).

Lemma 2.4. *The Cauchy operator defined by* (2.9) *is compact.*

3. Existence results

First, we introduce the following definition of the mild solution to our problem, based on the representation (2.8).

Definition 3.1. *For given* $\xi \in C([-h,0]; L^2(\Omega))$, *a function* $x \in C([-h,T]; L^2(\Omega))$ *is called a mild solution to* (1.1)-(1.3) *on the interval* $[-h,T]$ *iff* $x(\cdot,s) = \xi(\cdot,s)$ *for* $s \in [-h,0]$ *and*

$$x(\cdot,t) = S(t)\xi(\cdot,0) + \int_0^t S(t-s) f(s, x_\rho(\cdot,s)) ds, t \in [0,T].$$

We denote $C_\xi([0,T]; L^2(\Omega)) = \{x \in C([0,T]; L^2(\Omega)) : x(\cdot,0) = \xi(\cdot,0)\}$ for given $\xi \in C([-h,0]; L^2(\Omega))$. For $x \in C_\xi([0,T]; L^2(\Omega))$, we define $x[\xi] \in C([-h,T]; L^2(\Omega))$ as follows

$$x[\xi](\cdot,t) = \begin{cases} x(\cdot,t) & \text{if } t \in [0,T], \\ \xi(\cdot,t) & \text{if } t \in [-h,0]. \end{cases}$$

Hence, we have

$$x[\xi]_\rho(\cdot,t)=\begin{cases} x(\cdot,t-\rho(t)) & \text{if } t-\rho(t)\in[0,T],\\ \xi(\cdot,t-\rho(t)) & \text{if } t-\rho(t)\in[-h,0].\end{cases}$$

Let $\Phi: C_\xi([0,T];L^2(\Omega))\to C_\xi([0,T];L^2(\Omega))$ be the operator defined by

$$\Phi(x)(\cdot,t)=S(t)\xi(\cdot,0)+\int_0^t S(t-s)f(s,x[\xi]_\rho(\cdot,s))ds,$$

which will be referred to as *the solution operator*. This operator is continuous if f is a continuous map. Obviously, $x[\xi]$ is a mild solution of (1.1)-(1.3) iff x is a fixed point of Φ.

In the next theorem, we show a global existence result for (1.1)-(1.3).

Theorem 3.1. *Let* $f:[0,T]\times L^2(\Omega)\to L^2(\Omega)$ *be a continuous mapping such that*

(F1) $\limsup\limits_{\|v\|\to 0}\dfrac{\|f(t,v)\|}{\|v\|}=p$ *with* $p\in[0,\lambda_1)$ *for all* $t\in[0,T]$.

Then there exists $\delta>0$ *such that the problem (1.1)-(1.3) has at least one mild solution on* $[-h,T]$ *for any* $T>0$, *provided that* $\|\xi\|_\infty<\delta$.

Proof. Put $A=\lambda_1^{-1}\sup\limits_{t\in[0,T]}(1-\omega(t,\lambda_1))$. By assumption, one can take $\epsilon>0$ such that $(p+\epsilon)A<1$, and there exists $\eta>0$ such that $f(t,v)\le(p+\epsilon)\|v\|$ for all $\|v\|\le 2\eta$. Let

$$\delta_0=\eta\inf_{t\in[0,T]}\left\{\left[\omega(t,\lambda_1))+(p+\epsilon)\lambda_1^{-1}(1-\omega(t,\lambda_1))\right]^{-1}\left[1-(p+\epsilon)\lambda_1^{-1}(1-\omega(t,\lambda_1))\right]\right\},$$

then $\delta_0>0$. Indeed, observing that $\omega(t,\lambda_1)+(p+\epsilon)\lambda_1^{-1}(1-\omega(t,\lambda_1))\le 1+(p+\epsilon)A$. It implies that $\left[\omega(t,\lambda_1)+(p+\epsilon)\lambda_1^{-1}(1-\omega(t,\lambda_1))\right]^{-1}\ge\left[1+(p+\epsilon)A\right]^{-1}$. Hence

$$\begin{aligned}\delta_0&\ge\eta\left[1+(p+\epsilon)A\right]^{-1}\inf_{t\in[0,T]}\left[1-(p+\epsilon)\lambda_1^{-1}(1-\omega(t,\lambda_1))\right]\\ &\ge\eta\left[1+(p+\epsilon)A\right]^{-1}\left[1-(p+\epsilon)\lambda_1^{-1}\sup_{t\in[0,T]}(1-\omega(t,\lambda_1))\right]>0.\end{aligned}$$

Let B_η be the closed ball in $C_\xi([0,T];L^2(\Omega))$ centered at origin with radius η. Considering $\Phi:B_\eta\to C_\xi([0,T];L^2(\Omega))$, we have

$$\|\Phi(x)(\cdot,t)\|\le\omega(t,\lambda_1)\|\xi(\cdot,0)\|+\int_0^t\omega(t-s,\lambda_1)\|f(s,x[\xi]_\rho(\cdot,s))\|ds,$$

thanks to Lemma 2.3. Put $\delta=\min\{\delta_0,\eta\}$. If $\xi\in C([-h,0],L^2(\Omega))$ such that $\|\xi\|_\infty<\delta$, then $\|x[\xi]_\rho(\cdot,s)\|\le\|x\|_\infty+\|\xi\|_\infty\le\eta+\delta\le 2\eta$, for all $s\in[0,T]$. So

$$\begin{aligned}\|\Phi(x)(\cdot,t)\|&\le\omega(t,\lambda_1)\|\xi\|_\infty+(p+\epsilon)\int_0^t\omega(t-s,\lambda_1)\|x[\xi]_\rho(\cdot,s)\|ds\\ &\le\omega(t,\lambda_1)\delta+(p+\epsilon)(\eta+\delta)\int_0^t\omega(t-s,\lambda_1)ds\\ &\le\omega(t,\lambda_1)\delta+(p+\epsilon)(\eta+\delta)(1-\omega(t,\lambda_1))\lambda_1^{-1}\\ &\le\delta\left[\omega(t,\lambda_1)+(p+\epsilon)\lambda_1^{-1}(1-\omega(t,\lambda_1))\right]+(p+\epsilon)\eta\lambda_1^{-1}(1-\omega(t,\lambda_1))\\ &\le\delta_0\left[\omega(t,\lambda_1)+(p+\epsilon)\lambda_1^{-1}(1-\omega(t,\lambda_1))\right]+(p+\epsilon)\eta\lambda_1^{-1}(1-\omega(t,\lambda_1))\le\eta,\forall t\in[0,T].\end{aligned}$$

We have shown that $\Phi(B_\eta)\subset B_\eta$ if $\|\xi\|_\infty<\delta$. Hence, for $\|\xi\|_\infty<\delta$, $\Phi:B_\eta\to B_\eta$ is a continous map. Employing the Schauder fixed point theorem, Φ has at least one fixed point. The proof is complete.

Theorem 3.2. *Let* $f:[0,T]\times L^2(\Omega)\to L^2(\Omega)$ *be a continuous mapping such that*

(F2) $\|f(t,v)\|\le a\|v\|+b$, *with* $a,b\ge 0$, *for all* $t\in[0,T]$.

Then the problem (1.1)-(1.3) *has at least one mild solution on* $[-h,T]$, *for any* $T>0$.

Proof.

Let $\psi\in C([0,T];\mathrm{R})$ be the unique solution of $\psi(t)=\|\xi\|_\infty+(a\|\xi\|_\infty+b)T+\int_0^t\psi(s)ds$ and $V=\{x\in C_\xi([0,T];L^2(\Omega)):\sup_{\tau\in[0,t]}\|x(\tau)\|\le\psi(t),\ \text{for all}\ t\in[0,T]\}$. Then V is a closed and convex subset of $C_\xi([0,T];L^2(\Omega))$. Since Φ is continuous and compact, it suffices to show that $\Phi(V)\subset V$. Let $x\in V$, then

$$\|\Phi(x)(\cdot,t)\|\le\|S(t)\|\|\xi(\cdot,0)\|+\int_0^t\|S(t-s)\|\|f(s,x[\xi]_\rho(\cdot,s))\|ds$$

$$\le\omega(t,\lambda_1)\|\xi\|_\infty+\int_0^t\omega(t,\lambda_1)(a\|x[\xi]_\rho(\cdot,s)\|+b)ds$$

$$\le\|\xi\|_\infty+\int_0^t\Big(a(\|\xi\|_\infty+\sup_{\tau\in[0,s]}\|x(\cdot,\tau)\|)+b\Big)ds$$

$$\le\|\xi\|_\infty+\Big(a\|\xi\|_\infty+b\Big)T+a\int_0^t\sup_{\tau\in[0,s]}\|x(\cdot,\tau)\|ds.$$

So we get $\sup_{\zeta\in[0,t]}\|\Phi(u)(\cdot,\zeta)\|\le\|\xi\|_\infty+\Big(a\|\xi\|_\infty+b\Big)T+a\int_0^t\psi(s)ds=\psi(t)$, which implies that $\Phi(x)\in V$. Applying the Schauder fixed point theorem, the proof is complete.

Theorem 3.3. *Let* $f:[0,T]\times L^2(\Omega)\to L^2(\Omega)$ *such that*

(F3) $f(\cdot,0)=0$ *and* $\|f(t,v_1)-f(t,v_2)\|\le k(r)\|v_1-v_2\|$ *for all* $t\in[0,T]$ *and* $v_1,v_2\in L^2(\Omega)$

such that $\|v_1\|,\|v_2\|\le r$, *where* $k(r)$ *is a nonnegative function such that*

$\limsup_{r\to 0}k(r)=\eta\in[0,\lambda_1)$.

Then there exists $\delta>0$ *such that the problem* (1.1)-(1.3) *has a unique mild solution on* $[-h,T]$ *for any* $T>0$, *provided* $\|\xi\|_\infty\le\delta$.

Proof. Observe that, the assumption implies $\|f(t,v)\|\le k(r)\|v\|$, for all $t\in[0,T]$, $\|v\|\le r$. It implies that $\limsup_{\|v\|\to 0}\frac{\|f(t,v)\|}{\|v\|}\le\limsup_{r\to 0}k(r)=\eta\in[0,\lambda_1)$.

Hence by Theorem 3.1 the problem (1.1)-(1.3) has a mild solution. It remains to prove the uniqueness. Assume that $x_1[\xi],x_2[\xi]$ are solutions of problem (1.1)-(1.3). Let $r=\max\{\|x_1[\xi]\|_\infty,\|x_2[\xi]\|_\infty\}$, then

$$\|x_1(\cdot,t)-x_2(\cdot,t)\|\le\int_0^t\|S(t-s)\|\|f(s,x_1[\xi]_\rho(\cdot,s))-f(s,x_2[\xi]_\rho(\cdot,s))\|ds$$

$$\le\int_0^t\omega(t,\lambda_1)k(r)\|x_1[\xi]_\rho(\cdot,s)-x_2[\xi]_\rho(\cdot,s)\|ds$$

$$\le\int_0^t k(r)\sup_{\tau\in[0,s]}\|x_1(\cdot,\tau)-x_2(\cdot,\tau)\|ds,$$

due to the fact that $x_1(\cdot,s)=x_2(\cdot,s)$ for $s\in[-h,0]$. Observing that, the last integral is nondecreasing in t, we have

$$\sup_{\zeta\in[0,t]}\|x_1(\cdot,\zeta)-x_2(\cdot,\zeta)\|\le\int_0^t k(r)\sup_{\tau\in[0,s]}\|x_1(\cdot,\tau)-x_2(\cdot,\tau)\|ds, t\in[0,T],$$

which implies that $\|x_1(\cdot,t)-x_2(\cdot,t)\|=0$ due to the standard Gronwall inequality. The proof is complete.

4. Finite-time attractivity

The target of this section is to prove the finite-time attractivity of every solution, which can be completed by using Halanay-type inequality for the analysis.

Lemma 4.1. *Let $w:[-\tau,+\infty)\to R^+$ be a continuous function satisfying*

$$w(\theta)\le\omega(\theta,\lambda)w_0+\int_0^\theta\omega(\theta-s,\lambda)[k\sup_{z\in[s-\rho(s),s]}v(z)+\kappa(s)]ds,\ \theta>0, \tag{4.1}$$

$$w(s)=\ell(s), s\in[-\tau,0], \tag{4.2}$$

where $0<k<\lambda$, $\ell\in C([-\tau,0];R^+)$ and $\kappa\in L^1_{loc}(R^+)$ which is nondecreasing. Then

$$w(\theta)\ \le\frac{\lambda}{\lambda-k}\big[w_0+\int_0^\theta\omega(\theta-s,\lambda)\kappa(s)ds\big]+\frac{k}{\lambda}\sup_{s\in[-\tau,0]}\ell(s),\ \forall\theta>0. \tag{4.3}$$

Proof. We make use of the following result (see (Phong 2021)): if $w\in C([-\tau,\infty);\ \mathrm{R}^+)$ is a nonnegative function satisfying $w(\theta)\le a(\theta)+b\sup_{z\in[-\tau,\theta]}v(z)$, $\theta>0$ and $w(s)=\ell(s)$, $s\in[-\tau,0]$, where $a(\cdot)$ is a nondecreasing function and $b\in(0,1)$, then

$$w(\theta)\le(1-b)^{-1}a(\theta)+b\sup_{s\in[-\tau,0]}\ell(s),\ \forall\theta>0. \tag{4.4}$$

It follows from (4.1) that

$$\begin{aligned}w(\theta)&\le w_0+\omega(\cdot,\lambda)*\kappa(\theta)+k\sup_{z\in[-h,\theta]}v(z)\int_0^\theta\omega(\theta-s,\lambda)ds\\&\le w_0+\omega(\cdot,\lambda)*\kappa(\theta)+k\lambda^{-1}\sup_{z\in[-h,\theta]}v(z).\end{aligned}$$

Since $\kappa(\cdot)$ is nondecreasing, it is easily seen that the function $\omega(\cdot,\lambda)*\kappa$ is nondecreasing. Applying inequality (4.4) for $a(\cdot)=w_0+\omega(\cdot,\lambda)*\kappa$ and $b=k/\lambda$, we get (4.3) as desired.

Now we prove the main result of this section.

Theorem 4.1. *Let $f:[0,T]\times L^2(\Omega)\to L^2(\Omega)$ be a continuous mapping such that $p\|v-u\|\ge\|f(t,v)-f(t,u)\|$ for all $u,v\in L^2(\Omega)$ and the constant $p<r\lambda_1$, where r is the solution in $(0,1)$ of cubic equation $y^3-3y+1=0$. If x is the solution with respect to the initial datum ξ, then x is attractive on $[0,T]$ for all $\xi\in C([-h,0];L^2(\Omega))$.*

Proof. We call $\hat{x}$ is the solution with respect to the initial datum φ. We have

$$x(T)-\hat{x}(T)=S(T)(\xi(0)-\varphi(0))+\int_0^T S(T-s)[f(s,x[\xi]_\rho(s))-f(s,\hat{x}[\varphi]_\rho(s))]ds.$$

We estimate

$$\|x(T)-\hat{x}(T)\| \le \omega(T,\lambda_1)\|\xi(0)-\varphi(0)\| + \int_0^T \omega(T-s,\lambda_1)\|f(s,x[\xi]_\rho(s))-f(s,\hat{x}[\varphi]_\rho(s))\|ds$$

$$\le \omega(T,\lambda_1)\|\xi(0)-\varphi(0)\| + \int_0^T \omega(T-s,\lambda_1)p\|u[\xi]_\rho(s)-v[\varphi]_\rho(s)\|ds$$

$$\le \omega(T,\lambda_1)\|\xi(0)-\varphi(0)\| + \int_0^T \omega(T-s,\lambda_1)p\sup_{[-h,s]}\|x(\tau)-\hat{x}(\tau)\|ds. \quad (4.5)$$

Now we apply Lemma 4.1 to get

$$\|x(T)-\hat{x}(T)\| \le \frac{\lambda_1}{\lambda_1-p}\|\xi(0)-\varphi(0)\| + \frac{p}{\lambda_1}\sup_{s\in[-h,0]}\|\xi(s)-\varphi(s)\| \text{ for all } T>0. \quad (4.6)$$

Combining (4.5) and (4.6) we have

$$\|x(T)-\hat{x}(T)\| \le \omega(T,\lambda_1)\|\xi(0)-\varphi(0)\| + \int_0^T \omega(T-s,\lambda_1)p\sup_{[-h,0]}\|\xi(\tau)-\varphi(\tau)\|ds$$

$$+\int_0^T \omega(T-s,\lambda_1)p\sup_{[0,s]}\|x(\tau)-\hat{x}(\tau)\|ds$$

$$\le \omega(T,\lambda_1)\|\xi(0)-\varphi(0)\| + p\|\xi-\varphi\|_\infty(1-\omega(T,\lambda_1))\lambda_1^{-1}$$

$$+\int_0^T \omega(T-s,\lambda_1)p\Big(\frac{\lambda_1}{\lambda_1-p}\|\xi(0)-\varphi(0)\| + \frac{p}{\lambda_1}\|\xi-\varphi\|_\infty\Big)ds$$

$$\le \omega(T,\lambda_1)\|\xi(0)-\varphi(0)\| + p\|\xi-\varphi\|_\infty(1-\omega(T,\lambda_1))\lambda_1^{-1}$$

$$+p\Big(\frac{\lambda_1}{\lambda_1-p}\|\xi(0)-\varphi(0)\| + \frac{p}{\lambda_1}\|\xi-\varphi\|_\infty\Big)\frac{1-\omega(T,\lambda_1)}{\lambda_1}.$$

We obtain $\|x(T)-\hat{x}(T)\| < \|\xi-\varphi\|_\infty$ for all $T>0$ if

$$\Big[\omega(T,\lambda_1)+p\frac{1-\omega(T,\lambda_1)}{\lambda_1-p}\Big]\|\xi(0)-\varphi(0)\| < \Big[1-p\frac{1-\omega(T,\lambda_1)}{\lambda_1}\Big(1+\frac{p}{\lambda_1}\Big)\Big]\|\xi-\varphi\|_\infty. \quad (4.7)$$

The condition (4.7) is satisfied if: $\omega(T,\lambda_1)+p\frac{1-\omega(T,\lambda_1)}{\lambda_1-p} < 1-p\frac{1-\omega(T,\lambda_1)}{\lambda_1}\Big(1+\frac{p}{\lambda_1}\Big)$

Therefore $p\frac{1-\omega(T,\lambda_1)}{\lambda_1-p}+p\frac{1-\omega(T,\lambda_1)}{\lambda_1}\Big(1+\frac{p}{\lambda_1}\Big) < 1-\omega(T,\lambda_1)$

Hence $\frac{p}{\lambda_1-p}+\frac{p}{\lambda_1}\Big(1+\frac{p}{\lambda_1}\Big) < 1 \Leftrightarrow p^3-3\lambda_1^2p+\lambda_1^3>0.$ (4.8)

We assumed that $0<p/\lambda_1<r$ with r is the solution in $(0,1)$ of cubic equation $y^3-3y+1=0$, then the condition (4.8) is satisfied.

References

E.G. Bajlekova, (2001), *Fractional Evolution Equations in Banach Spaces.* PhD Thesis, Eindhoven University of Technology.

E. Bazhlekova, B. Jin, R. Lazarov, Z. Zhou, (2015), *An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid*, Nummer. Math. 131 no. 1, 1-31.

C.M. Chen, F. Liu, K. Burrage, Y. Chen, (2013), *Numerical methods of the variable-order Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivative*, IMA J. Appl. Math. 78 no. 5, 924-944.

C. Fetecau, M. Jamil, C. Fetecau, D. Vieru, (2009), *The Rayleigh-Stokes problem for an edge in a generalized Oldroyd-B fluid*, Z. Angew. Math. Phys. 60 , no. 5, 921-933.

A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, (2006), *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam.

D. Lan, 2022, *Regularity and stability analysis for semilinear generalized Rayleigh-Stokes equations*, Evol. Equ. Control Theory, 11(1): 259-282.

T. B. Ngoc, N. H. Luc, V. V. Au, N. H. Tuan, Y. Zhou, (2021), *Existence and regularity of inverse problem for the nonlinear fractional Rayleigh-Stokes equations*, Math. Methods Appl. Sci. 43, 2532-2558.

V. N. Phong, D. Lan, (2021), *Finite-time attractivity of solutions for a class of fractional differential inclusions with finite delay*, J. Pseudo-Differ. Oper. Appl. 12, no. 1, Paper No. 5, 18 pp.

F. Shen, W. Tan, Y. Zhao, Y. Masuoka, (2006), *The Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivative model*, Nonlinear Anal. Real World Appl. 7, no. 5, 1072-1080.

N.H. Tuan, Y. Zhou, T.N. Thach, N.H. Can, (2019), *Initial inverse problem for the nonlinear fractional Rayleigh-Stokes equation with random discrete data*, Commun. Nonlinear Sci. Numer. Simul. 78, 104873, 18 pp.